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SINGULAR SOBOLEV CONNECTIONS WITH HOLONOMY

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We consider local Sobolev connections on $SU(2)$ bundles over the complement, in R^4, of a smoothly embedded compact 2-manifold. Finite action implies that a holonomy condition is satisfied and we obtain an a priori estimate for the connection 1-form in terms of curvature and the flat connection carrying the holonomy. The a priori estimate classifies the possible singularities in these connections by the set of flat connections. In a certain case, this leads to smoothness and extendability results.

Let N be a full 4-dimensional neighborhood of the singular set S. The objects of study are connections $D = d + A$ defined on $SU(2)$ bundles over $X = N \backslash S$. We assume that $A \in H^2_{1,\mathrm{loc}}(X)$ and that the action is finite, i.e., the curvature $F = dA + A \wedge A$ is in $L^2(N)$.

The following *holonomy condition* was first stated by Cliff Taubes. Choose coordinates (r, θ, u, v) with (u, v) coordinates on S and (r, θ) coordinates in a plane normal to S. Fixing u and v, and denoting by A_θ the θ component of A, the initial value problem for an $SU(2)$ valued function,

$$\frac{dg_r}{d\theta} + A_\theta g_r = 0, \qquad g_r(0) = I,$$

has a unique solution $g_r(\theta)$, with $g_r(2\pi) = J_r \in SU(2)$. The holonomy condition we require is

$$\text{(H)} \qquad \lim_{r \to 0} J_r = J^\flat \text{ exists.}$$

This condition is *gauge invariant* up to conjugacy in $SU(2)$. Our results can be formulated in two theorems.

THEOREM 1. *If A and F are smooth on $N \backslash S$ and $F \in L^2(N)$, then* (H) *is satisfied for almost all u and v. Up to conjugacy, the limit is independent of u and v.*

Next, assume (H) holds. Locally, the conjugacy class $[J^\flat] \in SU(2)$ uniquely defines a flat connection $A^\flat = C\,d\theta$ with C a constant element of $su(2)$ determined up to a similarity transformation. Our second result uses holonomy to obtain an a priori estimate. We denote by X_0 and N_0 the intersections of X and N with a small open set in R^4 having nonvoid intersection with S.

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THEOREM 2. *Suppose* $\hat{D} = d + \hat{A}$ *satisfies* (H) *with* $\hat{A} \in H^2_{1,\mathrm{loc}}(X_0)$ *and* $\|F\|_{L^2(N_0)}$ *sufficiently small. Then there is a flat connection* $A^\flat$ *determined by* $[J^\flat]$, *and a universal constant* K, *such that* $\hat{D}$ *is gauge equivalent to* $D = d+A$, *with* $d^*A = 0$ *and*

$$\|A - A^\flat\|_{H^2_1(N_0)} \leq K\|F\|_{L^2(N_0)}.$$

Note that if $[J^\flat] = I$, then A is gauge equivalent to the zero connection form. In this case, D extends as an H^2_1 connection to all of N_0. If, in addition, field equations are satisfied, more smoothness follows from elliptic theory.

Theorem 1 is proved by making a good choice of gauge in which the Fourier coefficients of A_θ can be estimated in terms of F. These estimates can be used to show that $A_\theta\, d\theta$ converges to a flat connection as r tends to zero. This flat connection carries the holonomy. To show that the limit is independent of u and v requires another good choice of gauge and Stokes' theorem.

To prove Theorem 2, we carry out a plan of attack suggested by Cliff Taubes. This involves an open-closed argument similar to that used in [$\mathbf{U}_1$, Theorem 1.3]. The large space consists of the appropriate Sobolev space of connections satisfying the same holonomy condition. This space is shown to be connected. The subspace consists of connections which admit a Hodge gauge satisfying certain boundary and limiting conditions which imply the a priori estimate. (Detailed proofs will appear in a forthcoming article.)

Theorem 1 settles a conjecture of Atiyah's. Both theorems are related to recent work on the moduli space of magnetic monopoles over hyperbolic 3-space [**A**, **B**, **C**, **F**] and to Yang-Mills fields over S^4 whose topological charge is not integral [$\mathbf{FH}_1, \mathbf{FH}_2$].

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