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ON PROBLEMS OF U. SIMON CONCERNING MINIMAL SUBMANIFOLDS OF THE NEARLY KAEHLER 6-SPHERE

FRANKI DILLEN, LEOPOLD VERSTRAELEN AND LUC VRANCKEN

ABSTRACT. We classify the complete 3-dimensional totally real submanifolds with sectional curvature $K \geq \frac{1}{16}$ in the nearly Kaehler 6-sphere $S^6(1)$, and, as a corollary, we solve a problem for compact 3-dimensional totally real submanifolds of $S^6(1)$ related to U. Simon's conjecture for compact minimal surfaces in spheres.

1. The nearly Kaehler 6-sphere. It is well known that a 6-dimensional sphere S^6 does not admit any Kaehler structure, and whether S^6 does or does not admit a complex structure, as far as we know, is still an open question. However, using the Cayley algebra $\mathcal{C}$, a natural *almost complex structure* J can be defined on S^6 considered as a hypersurface in $\mathbf{R}^7$, which itself is viewed as the set $\mathcal{C}_+$ of the purely imaginary Cayley numbers (see, for instance, E. Calabi [**1**]). Together with the *standard metric* g on S^6, J determines a *nearly Kaehler structure* in the sense of A. Gray [**9**], i.e. one has $\forall X \in \mathcal{X}(S^6)$: $(\tilde{\nabla}_X J)(X) = 0$, where $\tilde{\nabla}$ is the Levi Civita connection of g. For reasons of normalization only, in the following we will always work with this nearly Kaehler structure on the sphere $S^6(1)$, of radius 1 and constant sectional curvature 1. The compact simple Lie group G_2 is the group of automorphisms of $\mathcal{C}$ and acts transitively on $S^6(1)$. Moreover, G_2 preserves both J and g.

2. Special submanifolds of $(S^6(1), g, J)$. With respect to J, two natural particular types of submanifolds M of $S^6(1)$ can be investigated: those which are *almost complex* (i.e. for which the tangent space of M at each point is invariant under the action of J) and those which are *totally real* (i.e. for

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which the tangent space of M at each point is mapped into the normal space by J). There only exist 2-dimensional almost complex submanifolds in $S^6(1)$, and these are always minimal [**10**]. Curvature properties for such surfaces were first obtained by K. Sekigawa [**15**]. Totally real submanifolds of $S^6(1)$ have either dimension 2 or 3. N. Ejiri [**7**] showed that every 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal, and he first obtained curvature properties for such submanifolds. The 3-dimensional totally real submanifolds of $S^6(1)$ were also considered by J. B. Lawson Jr. and R. Harvey [**11**] in their study of calibrated geometries and by K. Mashimo [**13**] in his classification of such compact submanifolds which are orbits of closed subgroups of G_2. In our study of submanifolds of the nearly Kaehler 6-sphere, we concentrated on the following problems.

PROBLEM A. Which real numbers can be realised as the constant sectional curvatures of almost complex or minimal totally real submanifolds M of $S^6(1)$?

PROBLEM B. Let K_1 and K_2 be two consecutive numbers as in Problem A. Then, do there exist compact submanifolds M of $S^6(1)$ whose sectional curvatures K satisfy $K_1 \leq K \leq K_2$, other than those for which $K \equiv K_1$ or $K \equiv K_2$?

3. On minimal submanifolds of arbitrary spheres. For minimal surfaces in a unit sphere $S^n(1)$ of arbitrary dimension n, one has a complete answer to Problem A (given by O. Boruvka, E. Calabi and N. Wallach for the case of positive Gauss curvature, the solutions being $K = 2/m(m+1)$, $m \in \mathbf{N}_0$, and by R. Bryant, proving the nonexistence of minimal surfaces of constant negative Gauss curvature in any sphere). Concerning Problem B, U. Simon conjectured the following.

CONJECTURE OF U. SIMON [**12**]. *Let M be a compact surface whose Gauss curvature K satisfies $2/m(m+1) \leq K \leq 2/m(m-1)$, for some $m \in \mathbf{N}\backslash\{0,1\}$, which is minimally immersed in $S^n(1)$. Then $K \equiv 2/m(m+1)$ or $K \equiv 2/m(m-1)$ (and hence M is a Boruvka sphere).*

For $m = 2$ and $m = 3$, this conjecture is known to be true, as was shown by H. B. Lawson Jr., U. Simon, M. Kothe, K.-D. Semmler, K. Benko and M. Kozlowski. Recently, quite a number of people have been working on this problem; in particular, T. Ogata, S. Montiel, T. Itoh, G. Jensen, M. Rigoli, J. Bolton, L. Woodward and U. Simon, A. Schwenk, B. Opozda together with the present authors. As far as we know however, in general this conjecture is still open for $m \geq 3$. In view of U. Simon's conjecture, we would like to call problems of type A and B, as stated above for almost complex and totally real submanifolds of $S^6(1)$, "*problems of U. Simon*".

4. Solutions of problems A and B.

THEOREM 1 [**15**]. *If an almost complex surface M in $S^6(1)$ has constant Gauss curvature K, then either $K = 1$ (and M is totally geodesic) or $K = \frac{1}{6}$ or $K = 0$.*

THEOREM 2 [**4, 2**]. *Let M be a compact almost complex surface in $S^6(1)$ with Gauss curvature K.*

(i) *Let* $\frac{1}{6} \leq K$ (≤ 1); *then either* $K \equiv \frac{1}{6}$ *or* $K \equiv 1$.
(ii) *If* $0 \leq K \leq \frac{1}{6}$, *then either* $K \equiv 0$ *or* $K \equiv \frac{1}{6}$.

THEOREM 3 [**6**]. *If a minimal totally real surface* M *in* $S^6(1)$ *has constant Gauss curvature* K, *then either* $K = 1$ (*and* M *is totally geodesic*) *or* $K = 0$.

THEOREM 4 [**6**]. *For a compact minimal totally real surface* M *in* $S^6(1)$ *with nonnegative Gauss curvature* K (*or equivalently* $0 \leq K \leq 1$), *either* $K \equiv 0$ *or* $K \equiv 1$.

In 1981, making use of a special choice of local orthonormal frames, N. Ejiri solved Problem A in the remaining case as follows.

THEOREM 5 [**7**]. *If a 3-dimensional totally real submanifold* M *of* $S^6(1)$ *has constant sectional curvature* K, *then either* $K = 1$ (*and* M *is totally geodesic*) *or* $K = \frac{1}{16}$.

Totally real 3-dimensional totally geodesic submanifolds in $S^6(1)$ are not hard to construct. On the other hand, N. Ejiri [**8**] proved that $S^3(\frac{1}{16})$ can be immersed totally real and isometrically in $S^6(1)$. K. Mashimo [**13**] found an orbit of a closed subgroup of G_2 with constant curvature $\frac{1}{16}$. Later we will explicitly describe these immersions, obtaining for instance as extra information that they are in fact 56-fold coverings of $S^3(\frac{1}{16})$. Compared to the solutions given in Theorems 2 and 4, the solution of Problem B seems more involved in the present case. In our approach, the solution represented by Theorem 6 is an immediate consequence of the Main Theorem.

THEOREM 6. *A compact 3-dimensional totally real submanifold of* $S^6(1)$ *whose sectional curvature* K *satisfies* $\frac{1}{16} \leq K \leq 1$ *has constant sectional curvature* $K = \frac{1}{16}$ *or* $K = 1$.

MAIN THEOREM. *Let* $x\colon M^3 \to S^6(1)$ *be a totally real isometric immersion of a complete 3-dimensional Riemannian manifold* M^3 *into the nearly Kaehler 6-sphere* $S^6(1)$. *If the sectional curvatures* K *of* M^3 *satisfy* $K \geq \frac{1}{16}$, *then either* M^3 *is simply connected and* x *is* G_2*-congruent to* $x_1\colon M_1 \to S^6(1)$ (*in which case* K *attains all values in the closed interval* $[\frac{1}{16}, \frac{21}{16}]$) *or to* $x_2\colon M_2 \to S^6(1)$ (*in which case* $K \equiv 1$), *or else* $\tilde{x}$, *the composition of the universal covering map of* M^3 *with* x, *is* G_2*-congruent to* $x_3\colon M_3 \to S^6(1)$ (*in which case* $K \equiv \frac{1}{16}$).

SKETCH OF PROOF (details will appear elsewhere [**3**]). As in our partial solution of Problem B [**5**], a crucial role is played by some integral formulas of A. Ros, of which we'll state one next. We do believe that these formulas provide a powerful tool to study problems in global Riemannian geometry.

LEMMA OF A. ROS [**14**]. *Let* M *be a compact Riemannian manifold,* UM *its unit tangent bundle and* UM_p *the fibre of* UM *over a point* p *of* M. *Let* dp, du *and* du_p *denote the canonical measures on* M, UM *and* UM_p, *respectively. For any continuous function* $f\colon UM \to \mathbf{R}$, *one has*

$$\int_{UM} f\,du = \int_M \left\{ \int_{UM_p} f\,du_p \right\} dp.$$

Now, let T be any k-covariant tensor field on M. Then one has the integral formula $\int_{UM}(\nabla T)(u,u,\dots,u)\,du = 0$, where ∇ is the Levi Civita connection on M.

We apply this lemma for some particular tensors T constructed in terms of the second fundamental form h of the immersion x. Then, under the assumption $K \geq \frac{1}{16}$, amongst others, we obtain that

$$R(v, A_{Jv}v; A_{Jv}v, v) = \frac{1}{16}\{\|A_{Jv}v\|^2 - \langle A_{Jv}v, v\rangle^2\}$$

for all $p \in M^3$ and all $v \in UM_p^3$, where R is the Riemann-Christoffel curvature tensor of M^3 and A is the Weingarten map with respect to a normal section ξ. From this, working with special frames, using the Gauss equation and with the help of computer manipulation of formulas, we can prove that at each point p the second fundamental form h_p has either one of three possible forms, leading respectively to the possibilities $K(p) \equiv 1$, $K(p) \equiv \frac{1}{16}$ and $K(p) \in [\frac{1}{16}, \frac{21}{16}]$, where $K(p)$ is the sectional curvature function of M^3 at p. In the following, we will give comments concerning only x_1 (x_2 is the totally geodesic case, and for x_3 we will confine ourselves to give precise formulas for the immersion). The existence of x_1 is guaranteed by the following result taken from a preprint by N. Ejiri.

THEOREM OF N. EJIRI [8]. *Let M be a 3-dimensional simply connected Riemannian manifold with metric $\langle\ ,\ \rangle$. Suppose there exist a $(1,2)$-symmetric tensor field T on M such that*

(i) $\operatorname{Tr} T = 0, \langle T(X,Y), Z\rangle = \langle T(X,Z), Y\rangle$,

(ii) $\langle R(X,Y)W,Z\rangle = \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle + \langle T(X,Z),T(Y,W)\rangle - \langle T(X,W),T(Y,Z)\rangle$,

(iii) $(\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z) + T(Z, X\wedge Y) = 0$, *where $\wedge$ is the vector product determined by some orientation on M.*

Then, up to a transformation of G_2, there exists a unique isometric immersion x of M into S^6 as a totally real submanifold with second fundamental form $J(x_*T)$ and with normal connection D defined by $D_X J(x_*Y) = J(x_*(\nabla_X Y + X\wedge Y))$.

Namely, on the unit sphere $S^3(1) = \{y = (y_1,y_2,y_3,y_4) \in \mathbf{R}^4 \mid \sum y_j^2 = 1\}$ we can define a metrix $\langle\ ,\ \rangle$, vector product $\wedge$ and tensor field T satisfying the conditions of this theorem and for which K attains all values in $[\frac{1}{16}, \frac{21}{16}]$. This leads to the immersion $x_1\colon S^3(1) \subset \mathbf{R}^4 \to S^6(1) \subset \mathbf{R}^7\colon y \to z = (z_1,\dots,z_7)$, where

$$z_1(y) = \frac{1}{9}(5y_1^2 + 5y_2^2 - 5y_3^2 - 5y_4^2 + 4y_1),$$

$$z_2(y) = -\frac{2}{3}y_2, \quad z_3(y) = \frac{2\sqrt{5}}{9}(y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_1),$$

$$z_4(y) = \frac{\sqrt{3}}{9\sqrt{2}}(-10y_1y_3 - 2y_3 - 10y_2y_4), \quad z_5(y) = \frac{\sqrt{15}}{9\sqrt{2}}(2y_1y_4 - 2y_4 - 2y_2y_3),$$

$$z_6(y) = \frac{\sqrt{15}}{9\sqrt{2}}(2y_1y_3 - 2y_3 + 2y_2y_4), \quad z_7(y) = -\frac{\sqrt{3}}{9\sqrt{2}}(10y_1y_4 + 2y_4 - 10y_2y_3).$$

In practice x_1 was found solving the system of differential equations (1) on p. 67 of M. Spivak's volume IV [**16**]; the rigidity of course follows from the fundamental theorem of submanifolds.

Finally, we mention the formulas of $x_3 \colon S^3(\frac{1}{16}) = \{y \in \mathbf{R}^4 \mid \sum y_j^2 = 16\} \subset \mathbf{R}^4 \to S^6(1) \subset \mathbf{R}^7 \colon y \mapsto z(y)$; we have

$$z_1(y) = \sqrt{15} \cdot 2^{-10} \cdot (y_1y_3 + y_2y_4) \cdot (y_1y_4 - y_2y_3)(y_1^2 + y_2^2 - y_3^2 - y_4^2),$$

$$z_2(y) = 2^{-12}\left[-\sum_j y_j^6 + 5\sum_{i<j} y_i^2y_j^2(y_i^2 + y_j^2) - 30\sum_{i<j<k} y_i^2y_j^2y_k^2\right],$$

$$\begin{aligned} z_3(y) = 2^{-10}[&y_3y_4(y_3^2 - y_4^2)(y_3^2 + y_4^2 - 5y_1^2 - 5y_2^2) \\ &+ y_1y_2(y_1^2 - y_2^2)(y_1^2 + y_2^2 - 5y_3^2 - 5y_4^2)], \end{aligned}$$

$$\begin{aligned} z_4(y) = 2^{-12}\{&y_2y_4(y_2^4 + 3y_3^2 - y_4^4 - 3y_1^4) + y_1y_3(y_3^4 + 3y_2^4 - y_1^4 - 3y_4^4) \\ &+ 2(y_1y_3 - y_2y_4)[y_1^2(y_2^2 + 4y_4^2) - y_3^2(y_4^2 + 4y_2^2)]\}, \end{aligned}$$

$$z_5(y_1, y_2, y_3, y_4) = z_4(y_2, -y_1, y_3, y_4),$$

$$\begin{aligned} z_6(y) = \sqrt{6} \cdot 2^{-12} \cdot [&y_1y_3(y_1^4 + 5y_2^4 - y_3^4 - 5y_4^4) - y_2y_4(y_2^4 + 5y_1^4 - y_4^4 - 5y_3^4) \\ &+ 10(y_1y_3 - y_2y_4)(y_3^2y_4^2 - y_1^2y_2^2)], \end{aligned}$$

$$z_7(y_1, y_2, y_3, y_4) = z_6(y_2, -y_1, y_3, y_4).$$

REFERENCES

1. E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geom. **1** (1967), 111-125.

2. F. Dillen, L. Verstraelen and L. Vrancken, *On almost complex surfaces of the nearly Kaehler* 6*-sphere*. II, Kōdai Math. J. **10** (1987), 261–271.

3. ——, *Classification of totally real* 3*-dimensional submanifolds of* $S^6(1)$ *with* $K \geq \frac{1}{16}$, preprint.

4. F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken, *On almost complex surfaces of the nearly Kaehler* 6*-sphere*. I, Collection of Scientific Papers, Faculty of Science, Univ. of Kragujevac **8** (1987), 5–13.

5. ——, *On totally real* 3*-dimensional submanifolds of the nearly Kaehler* 6*-sphere*, Proc. Amer. Math. Soc. **99** (1987), 741–749.

6. ——, *On totally real surfaces of the nearly Kaehler* 6*-sphere*, Geom. Dedicata (to appear).

7. N. Ejiri, *Totally real submanifolds in a* 6*-sphere*, Proc. Amer. Math. Soc. **83** (1981), 759–763.

8. ——, *Totally real submanifolds in a* 6*-sphere*, preprint.

9. A. Gray, *Minimal varieties and almost Hermitian submanifolds*, Michigan Math. J. **12** (1965), 273–287.

10. ——, *Almost complex submanifolds of the six sphere*, Proc. Amer. Math. Soc. **20** (1969), 277–279.

11. R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

12. M. Kozlowski and U. Simon, *Minimal immersions of 2-manifolds into spheres*, Math. Z. **186** (1984), 377–382.

13. K. Mashimo, *Homogeneous totally real submanifolds of* S^6, Tsukuba J. Math. **9** (1985), 185–202.

14. A. Ros, *A characterization of seven compact Kaehler submanifolds by holomorphic pinching*, Ann. of Math. (2) **121** (1985), 377–382.

15. K. Sekigawa, *Almost complex submanifolds of a 6-dimensional sphere*, Kōdai Math. J. **6** (1983), 174–185.

16. M. Spivak, *A comprehensive introduction to differential geometry*, vol. IV, Publish or Perish, Berkeley, 1979.

KATHOLIEKE UNIVERSITEIT LEUVEN, DEPARTEMENT WISKUNDE, CELESTIJNENLAAN 200B, 3030 LEUVEN, BELGIUM