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NUMERICAL METHODS FOR EXTREMAL PROBLEMS IN THE CALCULUS OF VARIATIONS AND OPTIMAL CONTROL THEORY

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Introduction. New, general methods are given to find numerical solutions for extremal problems in the calculus of variations and optimal control theory. Theoretical methods are derived and used to establish pointwise a priori error estimates with maximum error at the node point, $\|e\|_\infty$, equal to $O(h^2)$ and a Richardson error of $O(h^4)$. This is done under the weak assumption that there are no conjugate points on the interval and not under the usual convexity assumptions.

Of practical interest is that these methods (i) are very easy to implement, (ii) hold for well-defined mixtures of initial value and boundary value problems, (iii) use multipliers, and not ill-conditioned penalty methods, for both equality and inequality constraints, in a natural, efficient manner, and (iv) are applicable to transversality, type-minimal time problems.

The heart of these methods is the algorithm (4) and the a priori estimates in Theorem 2 for the m-dependent variable problem in the calculus of variations given below. Once this is established we quote Hestenes [**5**] and show that very general optimal control problems can be easily reformulated and solved as calculus of variations problems.

The calculus of variations problem. The problem is to find numerical solutions for extremal solutions of

$$I(x) = \int_a^b f(t, x, x')\, dt, \tag{1}$$

where $x(t)$ is an m-vector. This will be done by finding approximate numerical solutions of the first variational problem

$$I'(x, y) = \int_a^b (y^T f_x + y'^T f_{x'})\, dt = 0 \tag{2}$$

for numerical admissible variations $y(t)$. The setting and background is given in Hestenes [**5**, pp. 57-62]. In particular, we require that the $m \times m$ matrix $f_{x'x'}$ be invertible for each t in $[a, b]$, enough smoothness on f to yield a unique piecewise smooth solution, and that (1) have no conjugate points in $[a, b]$.

Letting $\pi = (a = a_0 < a_1 < \cdots < a_N = b)$ be a partition of $[a, b]$, with $a_{k+1} - a_k = h = (b-a)/N$, and $z_k(t)$ the spline hat functions with $z_k(a_k) = 1$,

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$Z_k(t) = z_k(t)I_{m\times m}$, and letting

$$x_h(t) = \sum_{k=0}^{N} Z_k(t)C_k \quad \text{and} \quad y_h(t) = \sum_{k=0}^{N} Z_k(t)D_k \tag{3}$$

be, respectively, the numerical solution to our problem and the numerical admissible variation, and utilizing the linearity of $y(t)$ in (2), we have the algorithm

(4)

$$\begin{aligned} f_{x'}\left(a_{k-1}^*, \frac{x_k + x_{k-1}}{2}, \frac{x_k - x_{k-1}}{h}\right) &+ \frac{h}{2} f_x\left(a_{k-1}^*, \frac{x_k + x_{k-1}}{2}, \frac{x_k - x_{k-1}}{h}\right) \\ &- f_{x'}\left(a_k^*, \frac{x_k + x_{k+1}}{2}, \frac{x_{k+1} - x_k}{h}\right) \\ &+ \frac{h}{2} f_x\left(a_k^*, \frac{x_k + x_{k+1}}{2}, \frac{x_{k+1} - x_k}{h}\right) = 0 \end{aligned}$$

for $k = 1, 2, \ldots, N-1$. In the above $a_k^* = (a_k + a_{k+1})/2$ and $x_k = x_h(a_k)$ is the computed value of the solution $x(t)$ at a_k.

We note that (4) is a block tridiagonal system of equations which is easily solved in practice by Newton's method with the accuracy described in Theorem 2 below. For the two-point boundary value problem with $x(a) = x_a$ and $x(b) = x_b$, (4) is a system of $m(N-1)$ nonlinear equations in $m(N-1)$ unknowns. For the initial value problem with $x(a) = x_a$ and $x'(a) = x'_a$, we have a nonlinear equation in the m variables x_{k+1} for each $k = 1, 2, \ldots, N-1$.

The first theorem involves long, but elementary calculations with local truncation error, see [**4**].

THEOREM 1. *Between corners of $x(t)$, the local truncation error is $h^3Q(a_k) + O(h^5)$, for $k = 1, 2, \ldots, N-1$.*

The vector $Q(t)$ depends only on the solution $x(t)$ and its derivatives and f and its derivatives. Thus,

THEOREM 2. *For $h > 0$ sufficiently small there exists $C > 0$ independent of h so that for any component e of the error $e_h(a_k) = x(a_k) - x_h(a_k)$ we have $|e| \leq Ch^2$. In addition, the Richardson solution $x_h^R(t)$, where*

$$x_h^R(a_k) = [4x_{h/2}(z_k) - x_h(a_k)]/3,$$

has a maximum component, pointwise error satisfying $|e^R| \leq Ch^4$, where e^R is any component of $e_h^R(a_k) = x(a_k) - x_h^R(a_k)$.

The proof of this result is too long and difficult to be given here and will appear elsewhere. A brief sketch is as follows. Using Theorem 1 and (4) we obtain an (approximate) second variational problem. This is a linear system $A_hE_h = h^3Q + O(h^5)$, where A_h is a block tridiagonal matrix, E_h is the $m(N-1)$ error vector, and $Q(a_k)$ is the kth component of Q, described above. Extensions of the author's quadratic form theory [**1**] and generalizations of results for ordinary differential equations by the author and Zeman [**3**] lead to an error of the form $\|E_h\|_2 \leq Ch^{3/2}$. Using these results it may be shown that the matrix A_h is invertible with bounded elements. Thus, $E_h = A_h^{-1}(h^3Q + O(h^5))$ implies that $\|E_h\|_\infty \leq Ch^2$. We note that this last result is a significant generalization of the classical result of Henrici [4] for $J = \operatorname{diag}(-1, 2, -1)$ when $m = 1$, where the elements of J^{-1} may be computed explicitly.

Optimal control problems. Our final result is to indicate that the practical and theoretical results obtained for the general calculus of variations problems above are applicable to a very large class of numerical optimal control problems.

Hestenes [**5**, pp. 346–351] shows that "A General Control Problem of Bolza" defined by the conditions

$$(5) \qquad I_p(x) = g_p(b) + \int_{t^0}^{t^1} L_p(t, x(t), u(t))\, dt \qquad (p = 0, 1, \dots, p),$$

$$(6) \quad \varphi_\alpha(t, x, u) \leq 0 \quad (1 \leq \alpha \leq m'), \qquad \varphi_\alpha(t, x, u) = 0 \quad (m' < \alpha \leq m),$$

$$(7) \qquad \dot{x}^i = f^i(t, x, u), \text{ and}$$

$$(8a) \qquad t^s = T^s(b), \qquad x^i(t^s) = X^{is}(b) \qquad (i = 1, \dots, n; s = 0, 1),$$

$$(8b) \qquad I_\gamma(x) \leq 0 \quad (1 \leq \gamma \leq p'), \qquad I_\gamma(x) = 0 \quad (p' < \gamma \leq p),$$

has a minimizing solution for $I_0(x)$ of the form

$$x_0 \colon x_0(t),\ b_0,\ u_0(t) \qquad (t^0 \leq t \leq t')$$

if there exist multipliers

$$\lambda_0 \geq 0,\ \lambda_\gamma,\ p_i(t),\ \mu_\alpha(t)$$
$$(\gamma = 1, \dots, p;\ i = 1, \dots, n;\ \alpha = 1, \dots, m),$$

not vanishing simultaneously, and functions

$$(9) \qquad H(t, x, u, p, \mu) = p_i f^i - \lambda_p L_p - \mu_\alpha \varphi_\alpha, \quad G(b) = \lambda_p g_p$$
$$(p = 0, 1, \dots, p)$$

satisfying the usual, expected conditions (see [**5**, pp. 348–350]).

Finally, we claim that

THEOREM 3. *The definitions of* $x^{n+1}(t), \dots, x^{n+q+m}(t)$ *given by*

$$(10a) \qquad \dot{x}^i = u^{i-n}, \qquad x^i(a) = 0 \qquad (i = n+1, \dots, n+q)$$

$$(10b) \quad \dot{x}^i = \mu_\alpha^{i-n-q}, \qquad x^i(a) = 0 \qquad (i = n+q+1, \dots, n+q+m)$$

allow us to convert the general control problem of Bolza to a problem of the form (1) *that admits a numerical solution with the errors described in Theorem 2 above.*

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