Essentially, it may be said that the authors consider two related types of problems which have been of interest in recent research. The first is to find "domination" theorems. Thus if $S, T\colon X \to Y$ are positive operators and $S \leqslant T$ then one seeks to determine properties of $S$ from those of $T$. This line was initiated in an important paper of Dodds and Fremlin in 1979, who showed that if $Y$ and the dual of $X$ have order-continuous norms and if $T$ is compact then $S$ is also compact. Since this paper there have been a number of ramifications of this general theme. The underlying idea is to prove, under suitable hypotheses, that $S$ can be approximated by operators in the ideal generated by $T$.

The second type of question revolves around factorization theorems. A very useful result of Davis, Figiel, Johnson, and Pełczyński in general Banach space theory asserts that a weakly compact operator between two Banach spaces can be factored through a reflexive space. The analogue for positive operators and Banach lattices is false, as was shown by a recent counterexample due to Talagrand. Unfortunately, Talagrand's example is not presented in the book (although it is mentioned), perhaps because it appeared too late for inclusion. Nevertheless, there are a number of factorization results available and the authors emphasize their use. A typical example is a result, due to Aliprantis and Burkinshaw, that a product of two positive weakly compact operators can be factored positively through a reflexive Banach lattice. Theorems of this type and their relatives can be used to establish domination-type results.

The cycle of ideas represented in the final two chapters closely follows the research interests of the authors over the last few years, and many of the results are due to them. It seems to the reviewer that these problems are now well understood, and most of the results are in the best possible form.

In general, this is a careful and well-written account of certain aspects of positive operators. It is clearly not and was not intended to be a complete treatment; the reader is given an introduction to some specific parts of the general theory. For a more complete understanding of all the current trends one should also consult the works of Schaefer and Zaanen.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987
©1987 American Mathematical Society
0273-0979/87 $1.00 + $.25 per page

*On the Cauchy problem*, by Sigeru Mizohata, Science Press, Beijing, and Academic Press, Orlando, 1985, 177 pp., $36.00 (paper). ISBN 0-12-501660-3

The theory of partial differential equations has as its source the study of a few model problems, many of them arising in physical applications. Laplace's equation and the wave and heat equations are prototypical, and the traditional (and appropriate) problems are Dirichlet's for the first and Cauchy's for the latter two. The techniques which have been developed over the years for handling general classes of equations have to a great extent been designed to give something approaching a description of the behavior of solutions as complete as that known for these special problems.

In his influential lectures at Yale [**12**], Hadamard formulated the notion of a well-posed problem for a partial differential equation. Three requirements are made: that the equation with its accompanying side conditions admit a solution on the region in question, that the solution be unique, and that the solution vary in some sort of continuous fashion when the side conditions are changed. An example of the Cauchy problem for an equation of evolution type may be described as follows. Let $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, and find $u(x, t)$ satisfying

$$(1)\qquad \left(\frac{\partial}{\partial t}\right)^m u(x,t) + \sum_{j=1}^{m} a_j\left(x, t; \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right)^{m-j} u(x,t) = f(x,t),$$

$$(2)\qquad \left.\left(\frac{\partial}{\partial t}\right)^j u(x,t)\right|_{t=0} = u_j(x), \qquad 0 \leqslant j \leqslant m-1.$$

The Cauchy problems for the wave equation

$$\left(\frac{\partial}{\partial t}\right)^2 u(x,t) - \left(\frac{\partial}{\partial x}\right)^2 u(x,t) = 0$$

and the heat equation

$$\left(\frac{\partial}{\partial t}\right) u(x,t) - \left(\frac{\partial}{\partial x}\right)^2 u(x,t) = 0$$

are well posed, the former for all time $t$ and the latter for time $t \geqslant 0$. But the Cauchy problem for Laplace's equation

$$\left(\frac{\partial}{\partial t}\right)^2 u(x,t) + \left(\frac{\partial}{\partial x}\right)^2 u(x,t) = 0$$

is not well posed, even in time $t \geqslant 0$. Indeed, let $u_0(x)$ be a $C^\infty$ function which is not analytic, and choose Cauchy data $u(x, 0) = u_0(x)$, $(\partial u/\partial t)(x, 0) \equiv 0$. Then if $u$ were a solution in $t \geqslant 0$, extension of $u$ via $u(x, -t) = u(x, t)$ would provide a solution on a neighborhood of $t = 0$ which would perforce be real analytic, in particular at $t = 0$. Thus no such solution exists. On the other hand, a solution of Laplace's equation with Cauchy data $u(x, 0) \equiv 0$, $(\partial u/\partial t)(x, 0) = \exp(-\sqrt{n})\sin(nx)$ does exist and is given by the expression

$$u(x,t) = \frac{\exp(-\sqrt{n})}{n} \sin(nx)\sinh(nt).$$

As $n \to \infty$, the data approach zero in any reasonable manner of measurement, while the solution does not.

Hadamard pointed out that continuous dependence on the data is no less important than the other two requirements, and in some sense should be forced on us by a correct view of the other two. Existence and uniqueness mean that a solution is known if the auxiliary data are known.

> But, in any concrete application, "known," of course, signifies "known with a certain approximation," all kinds of errors being possible, provided their magnitude remains smaller than a certain quantity; and, on the other hand, we have seen that the mere replacing of the value zero for $u_1$ by the (however small) value [of the Cauchy datum above], changes the solution not by very small but by very great quantities. Everything takes place, physically speaking, as if the knowledge of Cauchy's data would *not* determine the unknown function.

If the functions $a_j$, $f$, and $u_j$ in (1) and (2) are analytic, then the Cauchy-Kowalevsky Theorem [**4**, **5**, **6**, **7**, **18**] guarantees the existence of an analytic solution $u$ at least locally in time, and that solution is unique even in the class of $C^\infty$ functions by Holmgren's theorem [**13**]. But the example above for Laplace's equation already shows that the Cauchy-Kowalevsky Theorem is not the last word. Let $H^\infty$ stand for the space of all $C^\infty$ functions which together with all their derivatives are in $L^2(\mathbf{R}^n)$. Then Petrowsky [**23**] determined a necessary and sufficient condition for the Cauchy problem for the equation (1) to be well posed for solutions $u$ as functions of $t$ with values in $H^\infty$, in the case where the coefficients depend only on $t$. In particular, if $\mathring{a}_j(t; i\xi)$ is the part of $a_j(t; i\xi)$ homogeneous in $\xi$ of highest degree, and $\lambda_i(t; \xi)$ are the roots of the characteristic equation

$$\lambda^m + \sum_{j=1}^{m} \mathring{a}_j(t; i\xi)\lambda^{m-j} = 0, \tag{3}$$

then a necessary condition for $H^\infty$ well-posedness in the future $t \geqslant 0$ is $\operatorname{Re}\lambda_i(t; \xi) \leqslant 0$ for all $(t, \xi)$, $1 \leqslant i \leqslant m$. (Compare with the examples of the wave, heat, and Laplace equations.) Lax [**19**] and Mizohata [**21**] showed that a necessary condition for $H^\infty$ well-posedness for $t$ near 0 when the coefficients are also allowed to depend (smoothly) on $x$, in the case of a "Kowalevskian" evolution operator (where for each $j$ the order of $a_j(t, x; \xi)$ is at most $j$) is that the characteristic roots $\lambda_j$ of the equation

$$\lambda^m + \sum_{j=1}^{m} \mathring{a}_j(t, x; \xi)\lambda^{m-j} = 0 \tag{4}$$

all be purely real.

Since the 19th century, a standard tool in the proof of results about partial differential equations with constant coefficients has been the Fourier transform. For example, if the coefficients $a_j$ in equation (1) are independent of $x$, taking the partial Fourier transform with respect to $x$ in (1) and (2) leaves an ordinary differential equation with side conditions, depending on the Fourier transform variable $\xi$ as a parameter. Petrowsky [**22**] used this technique to give a detailed description of solutions in the case of "strictly hyperbolic" equations, those for which the characteristic polynomial (4) has $m$ real roots, distinct for $\xi \neq 0$.

In the 1950s and 1960s, pseudodifferential operators were developed, for instance in the work of Calderón and Zygmund [**2**, **3**], Mihlin [**20**], Kohn and Nirenberg [**17**], and Hörmander [**14**], to extend more fully to the case of variable coefficient equations the usefulness of Fourier transform techniques. Recall that by the Fourier inversion formula, if $u(x)$ is an appropriate function or if the integral is interpreted suitably in the sense of distributions, the action of a partial differential operator $P = p(x, i^{-1}\partial/\partial x)$ may be expressed as

$$Pu(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi. \tag{5}$$

Since $p(x, \xi)$ is a polynomial in its second variable, it satisfies (locally in $x$) estimates of the form

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial \xi}\right)^{\alpha} p(x,\xi)\right| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}. \tag{6}$$

It turns out that it is not important that $p$ be a polynomial in $\xi$; instead only estimates of the type (6), or even weaker conditions still, are all that are needed to make sense of expressions of the form (5) after integration by parts. Such symbols then correspond to operators which are, for example, bounded from the Sobolev space $H^s_{\text{compact}}$ to $H^{s-m}_{\text{local}}$. (The Sobolev space $H^s$ is defined as the space of distributions $u$ satisfying $(1+|\xi|)^s\hat{u}(\xi) \in L^2(\mathbf{R}^n)$.) In particular, the inverse of an "elliptic" symbol (one for which $|p(x,\xi)| \geqslant C|\xi|^m$ for $|\xi|$ large) satisfies an estimate of the form (6) with $m$ replaced by $-m$ for $|\xi|$ large, and hence corresponds to an operator bounded from $H^s_{\text{compact}}$ to $H^{s+m}_{\text{local}}$. It follows that elliptic regularity results may be recovered in a simple fashion if it can be established that the procedure of inverting an operator may be achieved to a first approximation by inverting its symbol. A "symbolic calculus" for operators with symbols satisfying (6) or its generalizations does in fact hold, and allows the calculation of the symbols of operators corresponding to compositions and adjoints from the symbols of the original operators.

The greater algebraic flexibility afforded by such a symbolic calculus allowed the proof of many results for variable coefficient operators that formerly had been limited to the translation invariant case. For example, Calderón [**1**] first obtained general results on the uniqueness of solutions for equations with smooth coefficients, extending Holmgren's Theorem, for the case in which the roots of the characteristic equation (4) are nonmultiple, by a factorization of the corresponding symbol. In the theorem of Lax and Mizohata on existence and uniqueness mentioned earlier, Lax dealt with the case of simple real roots in (4), while Mizohata removed the restriction on nonmultiplicity using a rudimentary form of the "microlocalization" of the problem.

The microlocalization procedure consists of using pseudodifferential operators to effect an appropriate localization in both the $x$ and the $\xi$ variables. It became well understood by the 1970s, when the notion of the "wave-front set" of a distribution was developed by Sato [**24**] and Hörmander [**15**]. Briefly, this set may be described as follows. Recall that by the Paley-Wiener-Schwartz Theorem, a distribution $u$ is in $C^\infty$ near a point $x_0$ if and only if, for any $\beta(x) \in C^\infty$ with support sufficiently near to $x_0$, the Fourier transform $(\beta u)^\wedge(\xi)$ is rapidly decreasing as $\xi \to \infty$. As a more refined notion, we say that $(x_0, \xi_0)$ is not in the wave-front set WF$(u)$ if for any $\beta(x) \in C^\infty$ with support sufficiently near to $x_0$ and $\alpha(\xi) \in C^\infty$ homogeneous of degree zero for large $\xi$ and supported on a sufficiently small conic neighborhood of the ray through $\xi_0$, $\alpha(\xi)(\beta u)^\wedge(\xi)$ is rapidly decreasing as $\xi \to \infty$. It follows that $u$ is in $C^\infty$ near $x_0$ if and only if $(x_0, \xi_0) \notin \mathrm{WF}(u)$ for any $\xi_0$, so that the wave-front set incorporates not only information about the singularities of $u$, but also about their Fourier spectrum.

The book of Mizohata under review assumes a basic familiarity with the standard pseudodifferential operators whose symbols satisfy estimates of the type (6). Since this theory and its generalizations have reached a certain stage of maturity, there are a variety of texts covering this material in great depth, for example the books of Treves [**26**], Taylor [**25**], Chazarain and Piriou [**8**], and the treatise of Hörmander [**16**]. A suitable companion volume in the present instance would be the very nice introduction by Folland [**9**]. Mizohata discusses the $H^\infty$ well-posedness of the Cauchy problem for (1) by the now standard technique of reduction to the case of a first-order pseudodifferential system. The necessary condition on the operator $\partial/\partial t - \lambda(x, t; \partial/\partial x)$ is demonstrated by using the familiar ideas of the pseudodifferential calculus, microlocalization, a commutator argument, and an energy estimate which is violated if data are chosen appropriately when the condition does not hold. Necessary and sufficient conditions for uniform $H^\infty$ well-posedness in the case of $\lambda(x, t; \xi)$ a first-order real-valued symbol are established, namely the "Levi conditions" on the lower-order terms.

The majority of the book is devoted to the analysis of the Cauchy problem in the Gevrey class $\gamma^s$ rather than in $H^\infty$. Gevrey [**10, 11**] initiated the study of such functions; $f$ is said to be in $\gamma^s$, $s > 1$, if locally it satisfies the estimate $|(\partial/\partial x)^\alpha f(x)| \leqslant A\alpha!^s C^{|\alpha|}$ for all $\alpha \geqslant 0$ or, equivalently, $f \in \gamma^s \cap C_0^\infty$ if $|f^\wedge(\xi)| \leqslant B \exp(-\varepsilon|\xi|^{1/s})$. Note that when this estimate holds for $s = 1$, $f$ is analytic. A typical example of a function of Gevrey class is $f(x) = \exp(-1/x)$, $x > 0$, $f(x) \equiv 0$, $x \leqslant 0$, which belongs to $\gamma^s$, $s \geqslant 2$. Such functions were analyzed in the context of the Cauchy problem for the heat equation with the roles of time and space reversed:

$$\begin{gathered}\left(\frac{\partial}{\partial t}\right)^2 u(x,t) - \left(\frac{\partial}{\partial x}\right) u(x,t) = 0, \\ u(x,0) = u_0, \qquad \frac{\partial u}{\partial t}(x,0) = u_1.\end{gathered} \tag{7}$$

Although not well posed in $H^\infty$, this problem is well posed in $\gamma^2$. Mizohata discusses the extension of this result to the class of weakly hyperbolic operators —those for which the principal symbol (4) has a factorization into real roots of constant multiplicity. When there are roots of multiplicity greater than one, well-posedness in $H^\infty$ will not in general hold without assumptions on the lower-order terms, but there will be unique solvability in some Gevrey class $\gamma^s$ without any further assumption. Mizohata notes that it seems that research on questions in this category is viewed rather negatively by some, though the French and the Soviets in particular have had a positive view. Perhaps the source of the negative opinions is already apparent in the problem (7): this expression is not the physically natural one to consider for the heat equation. Hadamard felt that

> it is remarkable, on the other hand, that a sure guide is found in physical interpretation: an analytical problem always being correctly set, in our use of the phrase, when it is the translation of some mechanical or physical question; and we have seen this to be the case for Cauchy's problem in the instances quoted in the first place.

The proofs given in the Gevrey class involve a factorization of the operator and a demonstration that the corresponding first-order pseudodifferential case can be solved by successive approximations which can be added up because of appropriate Gevrey estimates for solutions. Thus it becomes necessary to develop a pseudodifferential calculus for problems in this class; for example, the estimate (6) on symbols is replaced by

$$(8) \qquad \left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial \xi}\right)^\alpha p(x,\xi)\right| \leqslant C\alpha!\beta!^s C_0^{|\beta|}(1+|\xi|)^{m-|\alpha|} \quad \text{for } |\alpha| \leqslant 2n.$$

The calculus of such symbols and the corresponding operators is established; since the constants are all-important in (8), the results are somewhat delicate and the proofs somewhat tedious compared to their analogues in the $C^\infty$ category. Mizohata includes a representative but not exhaustive sample of such computations. He goes on to prove the analogue in Gevrey class of Hörmander's result **[15]** on the microlocal analysis of the singularities to solutions of strictly hyperbolic equations. Consider the first-order pseudodifferential equation

$$(9) \qquad \frac{\partial u}{\partial t} - \lambda\left(x, t; \frac{\partial}{\partial x}\right)u + c\left(x, t; \frac{\partial}{\partial x}\right)u = 0.$$

Let $(x(t), \xi(t))$ denote the null bicharacteristic strip passing through the characteristic point $(x_0, \xi_0)$ at time $t = 0$, that is,

$$\frac{dx}{dt} = \lambda_\xi(x, t; \xi), \qquad \frac{d\xi}{dt} = -\lambda_x(x, t; \xi), \qquad (x(0), \xi(0)) = (x_0, \xi_0).$$

Suppose that $\rho = \operatorname{order}\{c(x, t; \xi)\} \leqslant 0$. Hörmander's Theorem states that if $(x_0, \xi_0) \notin \mathrm{WF}(u)$, then $(x(t), \xi(t)) \notin \mathrm{WF}(u)$. In the Gevrey class, the definition of wave-front set is made corresponding to that in the $C^\infty$ case: $(x_0, \xi_0) \notin \mathrm{WF}_s(u)$ if for a $\gamma^s$ microlocalizer $\alpha(\xi)\beta(x)$ around $(x_0, \xi_0)$, $|\alpha(\xi)(\beta u)^\wedge(\xi)| \leqslant B \exp(-\varepsilon|\xi|^{1/s})$. Now suppose that $0 \leqslant \rho \leqslant 1/s$. The propagation of singularities theorem in Gevrey class states that if $(x_0, \xi_0) \notin \mathrm{WF}_s(u)$, then $(x(t), \xi(t)) \notin \mathrm{WF}_s(u)$. The analogue for higher-order differential operators is also given, again via a reduction to a first-order pseudodifferential system using the Gevrey calculus.

The book concludes with a brief discussion of problems modeled on the Schrödinger equation

$$i\left(\frac{\partial}{\partial t}\right)u(x,t)+\left(\frac{\partial}{\partial x}\right)^2 u(x,t)+c(x)u(x,t)=0, \tag{10}$$

rather than on the wave or heat equation. When first-order terms in $x$ are present, so that the equation becomes

$$\begin{aligned} i\left(\frac{\partial}{\partial t}\right)u(x,t)&+\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2 u(x,t)\\ &+\sum_{j=1}^{n} b_j(x)\left(\frac{\partial}{\partial x_j}\right)u(x,t)+c(x)u(x,t)=0, \end{aligned} \tag{11}$$

a necessary condition for well-posedness of the Cauchy problem in $L^2$ is established. A sufficient condition is proved by showing that under the appropriate assumption, (11) may be reduced to (10) by the pseudodifferential calculus. Current research is discussed, and further lines of inquiry are suggested.

Mizohata's lectures provide a readable introduction to several of the areas of modern research on aspects of the Cauchy problem. The book is accessible to graduate students who have already had a good introduction to the theory of pseudodifferential operators. For most of the results, either a complete proof is provided or an outline is given and a typical technical lemma is proved. I highly recommend the book to anyone interested in an introduction to Gevrey theory. It is disappointing, however, that the publisher did not have the manuscript revised to put this otherwise enjoyable account into grammatical English.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987
©1987 American Mathematical Society
0273-0979/87 $1.00 + $.25 per page

*Nonlinear approximation theory*, by Dietrich Braess, Springer Series in Computational Mathematics, vol. 7, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1986, xiv + 290 pp., $69.50. ISBN 0-387-13625-8

Approximation theory arose out of the need to represent "difficult" functions by "simpler" functions, precision being then traded for ease of computation. The theory concerns itself more with *classes* of functions than with individual functions. A central problem of perennial interest starts with a prescribed set $M$ in a normed space $E$. One contemplates the approximation of an element $f$ in $E$ by an element of $M$. The least error possible in this process is $d(f, M)$, defined to be the infimum of $\|f - m\|$ as $m$ ranges over $M$. If an element $m$ has the property $\|f - m\| = d(f, M)$, it is called a "best approximant" or "nearest point." Basic questions then are whether a nearest point exists, and if it does, whether it is unique, how it is to be recognized, and how it is to be computed. When a sequence of subsets $M_n$ is given, interesting *asymptotic* questions arise, such as whether the sequence $d(f, M_n)$ converges to zero and if so how rapidly.