

BMO ON THE BERGMAN SPACES OF THE CLASSICAL DOMAINS

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Let Ω be a bounded symmetric (Cartan) domain with its Harish-Chandra realization in \mathbf{C}^n [T]. For dv the usual Euclidean volume measure on $\mathbf{C}^n = \mathbf{R}^{2n}$, normalized so that $v(\Omega) = 1$, we consider the Hilbert space of square-integrable complex-valued functions $L^2 = L^2(\Omega, dv)$ and the Bergman subspace $H^2 = H^2(\Omega)$ of holomorphic functions in L^2 . The self-adjoint projection from L^2 onto H^2 is denoted by P . For f, g in L^2 , we consider the *multiplication operator* M_f on L^2 given by $M_f g = fg$ and the *Hankel operator* H_f on L^2 given by $H_f = (I - P)M_f P$. For f in L^2 , these operators are only densely defined and may be unbounded. The commutator $[M_f, P] = M_f P - P M_f$ is densely defined on L^2 and may also be unbounded. From the equations

$$[M_f, P] = H_f - H_{\bar{f}}^*, \quad (I - P)[M_f, P] = H_f, \quad [M_f, P](I - P) = -H_{\bar{f}}^*,$$

it follows that $[M_f, P]$ is a bounded operator if and only if $H_f, H_{\bar{f}}$ are bounded. Moreover, $[M_f, P]$ is a compact operator if and only if $H_f, H_{\bar{f}}$ are compact.

In earlier work [BCZ], it was shown that for f in $L^\infty(\Omega)$, the algebra of bounded measurable functions on Ω , $[M_f, P]$ is compact if and only if f has vanishing mean oscillation at the boundary $\partial\Omega$, where oscillation is defined in terms of the Bergman metric on Ω . In this note, we announce the companion result: *For f in L^2 , $[M_f, P]$ is bounded if and only if f is of "bounded mean oscillation on Ω ", where oscillation is defined as in [BCZ].* The space of such functions is denoted by $\text{BMO}(\Omega)$. We also obtain the expected result that: *For f in L^2 , $[M_f, P]$ is compact if and only if f is in the subspace $\text{VMO}_\partial(\Omega)$ of functions which have vanishing mean oscillation at the boundary $\partial\Omega$.* Our results are analogous to known results for arc-length measure on the unit circle [G, p. 278] and demonstrate the value of the Bergman metric in function-theoretic analysis on the classical domains.

Let $K(\cdot, a)$ be the Bergman reproducing kernel in $H^2(\Omega)$ for evaluation at $a \in \Omega$. For

$$k_a(\cdot) = K(a, a)^{-1/2} K(\cdot, a),$$

we define the *Berezin transform* of f in L^2 [BCZ] by

$$\tilde{f}(a) = \langle f k_a, k_a \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product. For typographical reasons, we write the Berezin transform of $|f|^2$ as $(|f|^2)^\sim$. It follows from known properties of

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the $\{k_a\}$ that \tilde{f} is defined and smooth (C^∞) everywhere on Ω . The transform \tilde{f} is critical to our analysis as are some previously unremarked properties of the Bergman metric $\beta(\cdot, \cdot)$ on Ω [K, p. 45; H, p. 298]. In the remainder of this note, we provide enough detail to rigorously state our main results and the main technical lemmas required for the proofs.

We recall that the Bergman metric $\beta(\cdot, \cdot)$ is a complete Riemannian metric on Ω which gives the usual topology on Ω [H, p. 52]. Moreover, the closed metric balls

$$E(a, r) = \{z: \beta(a, z) \leq r\}$$

are compact [H, p. 56]. By definition [K, p. 45], β is the “integrated form” of the infinitesimal metric

$$g_{ij}(z) = 1/2 \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z).$$

By $z \rightarrow \partial\Omega$, we mean that the usual distance function

$$d(z, \partial\Omega) \equiv \inf\{|z - w|: w \in \partial\Omega\}$$

has the property that $d(z, \partial\Omega) \rightarrow 0$. Let $BC(\Omega)$ denote the algebra of bounded continuous functions on Ω , with $C_\partial(\Omega)$ the subalgebra of all continuous functions for which $f(z) \rightarrow 0$ as $z \rightarrow \partial\Omega$. For f continuous on Ω , we define

$$\text{Osc}_z(f) = \sup\{|f(z) - f(w)|: \beta(z, w) \leq 1\}.$$

It is not hard to check, using the completeness of β , that $\text{Osc}_z(f)$ is also a continuous function of z . We say f is of *bounded oscillation* ($f \in \text{BO}(\Omega)$) if $\text{Osc}_z(f)$ is in $BC(\Omega)$ (as a function of z). We say f is of *vanishing oscillation at $\partial\Omega$* ($f \in \text{VO}_\partial(\Omega)$) if $\text{Osc}_z(f)$ is in $C_\partial(\Omega)$ (as a function of z) (cf. [BCZ]). For f in L^2 , the quantity

$$\text{MO}(f, z) = (|f|^2)^\sim(z) - |\tilde{f}(z)|^2$$

is a continuous function on Ω . We say f is of *bounded mean oscillation on Ω* ($f \in \text{BMO}(\Omega)$) if $\text{MO}(f, \cdot)$ is in $BC(\Omega)$. We say f has *vanishing mean oscillation at $\partial\Omega$* ($f \in \text{VMO}_\partial(\Omega)$) if $\text{MO}(f, \cdot)$ is in $C_\partial(\Omega)$.

We also have a more geometric notion of mean oscillation. Let $|E(z, r)| = v(E(z, r))$. For fixed $r > 0$ and f in L^2 , the quantities

$$\hat{f}(z, r) = |E(z, r)|^{-1} \int_{E(z, r)} f(w) dv(w),$$

$$\begin{aligned} \text{MO}_r(f, z) &= |E(z, r)|^{-1} \int_{E(z, r)} |f(w) - \hat{f}(z, r)|^2 dv(w) \\ &= \frac{1}{2} |E(z, r)|^{-2} \int_{E(z, r)} \int_{E(z, r)} |f(w) - f(u)|^2 dv(w) dv(u) \end{aligned}$$

are continuous functions on Ω . We say f in L^2 is in $\text{BMO}_r(\Omega)$ if $\text{MO}_r(f, \cdot)$ is in $BC(\Omega)$. We say f in L^2 is in $\text{VMO}_r^\partial(\Omega)$ if $\text{MO}_r(f, \cdot)$ is in $C_\partial(\Omega)$.

We require a few additional definitions. We write

$$\mathcal{F} = \{f \in L^2: (|f|^2)^\sim \in BC(\Omega)\}, \quad I = \{f \in L^2: (|f|^2)^\sim \in C_\partial(\Omega)\}.$$

Clearly, $\mathcal{F} \subset \text{BMO}(\Omega)$ and $I \subset \text{VMO}_\partial(\Omega)$. For S any subset of Ω , we write

$$\|f\|_{\text{BMO}(S)} \equiv \sup_{z \in S} \text{MO}(f, z)^{1/2}$$

and

$$\|f\|_{\text{BMO}} \equiv \|f\|_{\text{BMO}(\Omega)}, \quad \|f\|_r \equiv \sup_{z \in \Omega} \text{MO}_r(f, z)^{1/2}.$$

Our main result is

THEOREM A. *For f in L^2 , the following are equivalent:*

- (i) $H_f, H_{\bar{f}}$ are bounded.
- (ii) $f \in \text{BMO}(\Omega)$.
- (iii) $f \in \text{BMO}_r(\Omega)$ for all $r > 0$.
- (iv) $f \in \text{BMO}_r(\Omega)$ for some $r > 0$.
- (v) $f \in \text{BO} + \mathcal{F}$.

Moreover, the quantities $\max\{\|H_f\|, \|H_{\bar{f}}\|\}$, $\|[M_f, P]\|$, $\|f\|_{\text{BMO}}$, and $\|f\|_r$ are equivalent and \tilde{f} is in BO with $f - \tilde{f}$ in \mathcal{F} whenever any of (i)–(v) hold.

We also have a corresponding extension of the main result of [BCZ]:

THEOREM B. *For f in L^2 , the following are equivalent:*

- (i) $H_f, H_{\bar{f}}$ are compact.
- (ii) $f \in \text{VMO}_\partial(\Omega)$.
- (iii) $f \in \text{VMO}_\partial^r(\Omega)$ for all $r > 0$.
- (iv) $f \in \text{VMO}_\partial^r(\Omega)$ for some $r > 0$.
- (v) $f \in \text{VO}_\partial(\Omega) + I$.

Moreover, \tilde{f} is in $\text{VO}_\partial(\Omega)$ and $f - \tilde{f}$ is in I whenever any of (i)–(v) hold.

For x in \mathbb{C}^n and z in Ω we define

$$H(z, x) = \sum_{i=1}^n \sum_{j=1}^n x_i \bar{x}_j g_{ij}(z).$$

For f in H^2 and z in Ω we have the analytic gradient

$$\nabla_z f = \left(\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$$

and we can define, as in [T],

$$Q_f(z) = \sup\{|\langle \nabla_z f, \bar{x} \rangle| H(z, x)^{-1/2} : 0 \neq x \in \mathbb{C}^n\}.$$

Following [T], we say f is in the Bloch space $B(\Omega)$ if

$$\sup_{z \in \Omega} Q_f(z) \equiv \|f\|_B < \infty.$$

We say f is in the little Bloch space $B_0(\Omega)$ if

$$\lim_{z \rightarrow \partial\Omega} Q_f(z) = 0.$$

For f in H^2 , it is easy to see that $H_f = 0$. In this case, we obtain some additional information:

THEOREM C. *For f in H^2 , $H_{\bar{f}}$ is bounded if and only if f is in $B(\Omega)$. Moreover, $\|H_{\bar{f}}\|$ and $\|f\|_B$ are equivalent quantities.*

THEOREM D. For f in $H^2, H_{\bar{f}}$ is compact if and only if f is in $B_0(\Omega)$. For $\text{rank}(\Omega) \neq 1, B_0(\Omega)$ consists of just the constant functions.

The proofs of Theorems A and B depend on two key results about the Bergman metric and the Berezin transform.

THEOREM E. The function $\beta(0, \cdot)$ is in $L^p(\Omega, dv)$ for all $p > 0$.

THEOREM F. For any smooth curve $\gamma: I \rightarrow \Omega$ ($I = [0, 1]$) with $s = s(t)$ the arc-length of γ with respect to the Bergman metric $(g_{ij}(z))$, and for any f in $BMO(\Omega)$, we have

$$\left| \frac{d}{dt} \tilde{f}(\gamma(t)) \right| \leq 2\sqrt{2} \left(\frac{ds}{dt} \right) \|f\|_{BMO(\gamma(I))}.$$

We also use a recent result of [FK] and some estimates from [BCZ]. Proofs will appear elsewhere.

It should be recalled that, as $z \rightarrow \partial\Omega$,

$$\beta(0, z) \rightarrow +\infty, \quad K(z, z) \rightarrow +\infty.$$

The point of Theorem E is that, for bounded symmetric domains Ω , $\beta(0, z)$ does not blow up too badly near $\partial\Omega$.

REMARK. The function $\beta(0, \cdot)$ is the prototype of $BO(\Omega)$. Using the invariance of the metric $\beta(\cdot, \cdot)$ under automorphisms of Ω and Theorem E, it is easy to check that $\beta(0, \cdot)$ is in $BMO(\Omega)$. The function $\exp\{i\beta(0, \cdot)^{1/2}\}$ is in $VMO_{\partial}(\Omega)$, as noted in [BCZ].

The measure $dv(z)$ may be replaced by Harish-Chandra measures of the form $C_t K(z, z)^t dv(z)$ with $t < t_0(\Omega)$ and Theorems A and B remain true. We conjecture that Theorems A and B hold, with different proofs, for domains considerably more general than the bounded symmetric (Cartan) domains.

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