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An introduction to chaotic dynamical systems, by Robert L. Devaney, Benjamin/Cummings Publishing Company, Menlo Park, 1986, xii + 320 pp. ISBN 0-8053-1601-9

Normally, the Bulletin of the American Mathematical Society reviews research monographs and graduate texts. Rarely does it seem suitable that more elementary books be reviewed. In my opinion, the editors rightly felt that the book by Devaney is such an exception. The undergraduate curriculum in mathematics evolves rather slowly. Infrequently a serious attempt is made to introduce modern mathematics into the undergraduate curriculum. The reasons are clear. Most modern mathematics requires sufficient background preparation that it is simply not accessible to a typical undergraduate. Moreover, there is a question of priorities in choosing topics courses that are suitable for mathematics majors. Devaney has written a book which deserves serious consideration as a suitable undergraduate text for an advanced undergraduate topics course.

The main subject of Devaney's book, iteration of functions in one real and complex dimension, has grown to maturity in the past decade from humble and largely forgotten origins. It has flowered into an object of considerable mathematical (and visual) beauty and subtlety. Still, the basic phenomena of the real theory can be explained and understood with little background beyond the standard fare of calculus, linear algebra, and basic complex analysis. Additionally, the subject is one that is easy to illustrate, and it provides a wonderful arena for exploration with a computer. As a potential new item in the undergraduate curriculum, the importance of the subject and particularly the style of thought deserve extended discussion.

Iteration is a process that is frequently encountered in both the natural world and in artificial ones. An important example in an artificial world is to describe the behavior of Newton's method applied to a fixed function. If we believe that there are hard and fixed rules that determine the evolution of economies, populations, planets, or blocks sliding on inclined planes, then we confront the mathematical problem of making long-term predictions from our knowledge of how the rules operate for fixed short times. For continuous time, this is the problem of solving systems of ordinary differential equations. The usual undergraduate courses in differential equations begin with classical techniques for finding explicit solutions to systems of ordinary differential equations. The material has the flavor of "techniques of integration" in calculus and often is presented as a collection of tricks. One of the facts of life is that most differential equations do not have explicit analytic integrals, so the techniques one learns for solving equations explicitly are limited. Indeed, the presence of "chaos" in solutions is often an obstruction to the existence of analytic integrals. Dynamical systems theory seeks qualitative information about the solutions of general equations, often in a manner that looks at all of the phase space. This is an important problem that pervades much of science.

Mathematical tools that afford a new perspective on the problem deserve to be widely disseminated.

Chaotic solutions of differential equations do not occur for systems of first-order equations in one or two variables, and visualization of trajectories in three or more dimensions is difficult. By discretizing time, chaos can be found in two-dimensional diffeomorphisms. A further reduction to allow discrete systems to be noninvertible yields chaotic dynamics in maps of the interval. Having abstracted the problem of solving differential equations to arrive at a map of the interval, it is reasonable to ask whether the analysis of this simpler problem will lead to intuition and techniques that are useful in thinking about realistic problems. My answer to this question is definitely yes, but the reader should note that I have a vested interest in the field and see the world from its perspective.

What type of intuition can one develop by studying one-dimensional dynamics? First, one can learn very quickly that repeated application of simple deterministic rules can generate patterns of great complexity. This contrasts with the naive intuition that irregularity in a dynamical process is generated by the presence of noise or randomness. Second, one can learn that there are patterns in something that initially appears to be irregular. Much can be learned from a combinatorial analysis that requires few tools beyond the intermediate value theorem. For example, there is the striking theorem of Sharkovskii that gives an ordering of the positive integers $3 \gg 5 \gg 7 \gg \cdots \gg 16 \gg 8 \gg 4 \gg 2 \gg 1$ with the property that if a continuous map of the interval has a periodic orbit of a given period, then it has periodic orbits of all subsequent periods. The proof of this theorem is readily accessible from the intermediate value theorem. Third, one can learn many of the concepts that are used in dynamical systems theory to describe the qualitative properties of flows. The ideas of structural stability and hyperbolicity and how they are used to focus attention on typical phenomena can be explained. The mathematics here embodies a concern for describing hidden patterns in the real world that are ubiquitous but difficult to work with in direct quantitative calculations. Finally, the theory is one that is engrossing in its computational aspects. It is easy to write algorithms for studying systems numerically, and these algorithms produce surprising results and pretty pictures. Thus the subject is one that is an excellent choice for introducing undergraduates (and professional mathematicians!) to "experimental mathematics" done with a computer.

I have argued that dynamical systems theory is appropriate for an undergraduate mathematics course alongside topics courses in, say, topology or number theory. Devaney's book is thoroughly suitable as a text for such a course. The book could readily serve as a text for a second semester in an upper division course on ordinary differential equations. It is, however, the first text aimed at chaotic dynamical systems for undergraduates, so I expect that teaching a course from the book will require close attention and supervision. One should also be aware that the subject of the text is still new and evolving. This carries with it a sense of excitement, but it also means that there is less of a feeling that the material is etched for posterity in the granite of mathematical truth in its current form. If the subject finds its way into the curriculum of many colleges and universities, the next generation of texts may have a substantially different emphasis. Devaney's book is an excellent choice for professional mathematicians to read as an introduction to the subject. There are ample exercises.

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Markov processes: *Characterization and convergence*, by Stewart N. Ethier and Thomas G. Kurtz, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, and Singapore, 1986, x + 534 pp., $47.50. ISBN 0-471-08186-8

The theoretical side of mathematical probability has long been preoccupied with limit theorems; not surprisingly, since one of the natural interpretations of *probability* is as *long-run frequency*. Traditionally, limit theorems have been divided into two categories: strong limits, where some asymptotic event is asserted to occur with probability one, and weak limits, where the distributions of a sequence of random quantities are asserted to converge to a limit distribution. The prototype weak limit result is the *central limit theorem* (CLT), which says that under mild conditions the sums $S_n = X_1 + \cdots + X_n$ of independent random variables can asymptotically be approximated by Gaussian distributions. This is the key result in elementary mathematical statistics. The average height of a random sample of people is a random quantity whose exact distribution is very complicated, depending on the entire list of heights of the population; but the CLT says that the distribution of the average height of a large sample is approximately a Gaussian distribution with two parameters which depend only on the mean and standard deviation of the population heights. This illustrates the practical purpose of weak convergence theorems, to approximate complicated exact finite distributions by simpler limiting distributions. A more sophisticated example concerns neutral genetic models. Much observed genetic variation within a species (e.g., eye color in humans) confers no apparent selective advantage (i.e., is apparently "neutral"). Is it plausible that such variation is really neutral? To study this question one needs to set up a mathematical model and compare its predictions with observations. In detail, any model will be rather arbitrary and unrealistic, but one can hope that the long-term behavior of a model is insensitive to its details and instead approximates some mathematically natural process with only a few parameters.

Returning to the CLT, a pure mathematician would regard its proof as an easy exercise in Fourier analysis. Modern probabilists look at it differently. For each n, consider not only the single random variable S_n but instead the whole process $(S_m; 0 \leqslant m \leqslant n)$, which can be regarded as a random element of function space. Under the same conditions as the CLT these processes, suitably normalized, converge to the Brownian motion process $(B_t; 0 \leqslant t \leqslant 1)$. Not