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Theory of multipliers in spaces of differentiable functions, by V. G. Maz'ya and T. O. Shaposhnikova, Monographs and Studies in Mathematics, vol. 23, Pitman Publishing Co., Brooklyn, New York, 1985, xii + 344 pp., $49.95. ISBN 0-273-08638-3

1. Multipliers. One of the simplest examples of a multiplier in a space of differentiable functions is a measurable function $\gamma(x)$, $x \in \mathbf{R}^n$, such that the operator of pointwise multiplication $u \to \gamma \cdot u$ is bounded from the Sobolev space W_2^1 on $\mathbf{R}^n$ into L_2 on $\mathbf{R}^n$; equivalently, there is a constant c such that

$$(1) \qquad \int |\gamma(x) \cdot \phi(x)|^2 \, dx \leqslant c \int \left(|\nabla \phi(x)|^2 + |\phi(x)|^2\right) dx$$

for all $\phi \in C_0^\infty(\mathbf{R}^n)$. The space of all such γ is denoted by $M(W_2^1 \to L_2)$, with the smallest c in (1) the square of the multiplier norm of γ. Clearly, one can easily extend this notion to pairs of higher-order Sobolev spaces: $W_p^m \to W_q^k$, $k \leqslant m$, $1 \leqslant p, q < \infty$, or for that matter, to any of the various pairs of function spaces that naturally occur in analysis. The coefficients of a differential operator acting on Sobolev functions can be interpreted as multipliers. For example, if $P(x, D)u = \sum_{|\alpha| \leqslant k} a_\alpha(x) D_x^\alpha u$, then P: $W_p^m \to W_p^{m-k}$ is continuous when $a_\alpha \in M(W_p^{m-|\alpha|} \to W_p^{m-k})$. The function γ is called a compact multiplier if the operator of pointwise multiplication is a compact operator. The principal theme of the book under review (referred to below as *Multipliers*) is the characterization of multipliers and compact multipliers in the basic Sobolev-type spaces used in analysis. Because of their connection to differential equations, it is not surprising that there are plenty of sufficient conditions in the literature for multipliers or compact multipliers. For example, saying simply that "the coefficients are sufficiently smooth" in a differential operator is one such condition. The object, however, is to find necessary and sufficient conditions, or even just good sufficient conditions. Here the word "good" describes some ideal (largest) useful subclass of multipliers that can be easily and simply described.

A necessary and sufficient condition on γ that insures (1) for $n \geqslant 2$ is

$$\sup\left\{\int_E |\gamma(x)|^2\,dx/\operatorname{cap}(E)\right\} < \infty, \tag{2}$$

where the supremum is taken over all compact subsets E of $\mathbf{R}^n$ of diameter $\leqslant 1$. Here $\operatorname{cap}(E)$ is the infimum of the integral on the right-hand side of (1) over all $C_0^\infty(\mathbf{R}^n)$ functions ϕ that are $\geqslant 1$ on E. This set function (capacity) is an equivalent form of the classical Newtonian capacity of E when $n \geqslant 3$ and the logarithmic capacity of E when $n = 2$. When $n = 1$, condition (2) simplies. It becomes

$$\sup_x \int_{|x-y|<1} |\gamma(y)|^2\,dy < \infty. \tag{3}$$

These characterizations are the prototypes of the various characterizations of spaces of multipliers found in *Multipliers*. Condition (2) originated with V. G. Maz'ya [**1**]. The compact multiplier characterizations are given in an analogous fashion.

Condition (2) has its drawbacks. It requires that each compact set be checked; it is well known that just letting E in (2) range over all balls in $\mathbf{R}^n$ is not enough. Nevertheless, (2) is surprisingly useful. For example an easy potential-theoretic argument shows that

$$\sup_x \int_{|x-y|<1} |x-y|^{2-n}|\gamma(y)|^2\,dy < \infty \tag{4}$$

implies (2) when $n \geqslant 3$. Also the "isoperimetric inequality" between Newtonian capacity and Lebesgue measure of a body implies that if γ is in the Lebesgue space L_n, then (2) holds. Several authors have recently investigated the possibility of replacing condition (2) altogether, by a condition using only balls. For example, R. Kerman and E. Sawyer in [**2**] show that (1) is equivalent to

$$\sup_B \left\{\int_B \left[G_1\left(|\gamma|^2 \cdot X_B\right)\right]^2 dx \Big/ \int_B |\gamma|^2\,dx\right\} < \infty, \tag{5}$$

where X_B is the characteristic function of the ball B, and $G_1(f)$ denotes the Bessel potential of f of order 1. Also, for $n \geqslant 3$ there is a simple and elegant sufficient condition for (1) using only balls due to C. Fefferman and D. H. Phong [**3**]. It reads

$$\sup_B \left\{|B|^{2p/n-1}\int_B |\gamma(x)|^{2p}\,dx\right\} < \infty \tag{6}$$

for some $p > 1$. Notice that (6) reduces to (2) when $E = B$ and $p = 1$. Other recent work extending the Fefferman-Phong result are contained in [**4**, **5**, and **6**]. In these latter papers the idea is to characterize the multipliers on weighted Sobolev spaces.

2. Schrödinger operators. An area that has contributed to the theory of multipliers is the study of a particular differential operator, the Schrödinger operator: $H = -\Delta + V(x)$, Δ being the Laplace operator on $\mathbf{R}^n$ and $V(x)$ a potential function on $\mathbf{R}^n$. The idea over the years has been to find conditions on V that will yield a complete theory for such operators and still include all of the physically interesting potential functions. This theory starts simply with the problem of defining the operator and then studying its spectra, eigenvalues, eigenfunctions, etc. For example, asking that H be a bounded operator from W_2^2 into L_2 would require that V be in $M(W_2^2 \to L_2)$. This is rather severe. An alternative is to weaken the definition of H, allowing it to be only weakly defined on W_1^2—in physical language, defined as a "form sum". In any case, to guarantee that the domain is sufficiently large one needs the estimate

$$|(\phi, V\phi)| \leqslant a(\phi, -\Delta\phi) + b(\phi, \phi) \tag{7}$$

for all $\phi \in C_0^\infty(\mathbf{R}^n)$. The symbol $(\cdot,\cdot)$ denotes the usual L_2 inner product. Here a and b are positive constants with $a < 1$. If V has a fixed sign, then (7) clearly asks that at least it is $\pm$ the square of a multiplier in $M(W_2^1 \to L_2)$, one with a special size requirement on its norm. Thus a class of V that gives (7) with small a (at the expense of b) is of some interest in dealing with H. In the recent "comprehensive review" article [7], B. Simon singles out the good class ($n \geqslant 3$)

$$\lim_{\delta \to 0} \sup_x \int_{|x-y|<\delta} |x - y|^{2-n} |V(y)|\, dy = 0. \tag{8}$$

This condition plays a central role in [7]—as well as implying (7) for small a. Furthermore, (8) is very closely related to the compactness conditions for a multiplier γ, $V = |\gamma|^2$. In fact, if such a γ satisfies (8) and has compact support, then it will be a compact multiplier.

Condition (8) is the result of many years of study of the operators H. Early ideas for general conditions on V include those of T. Kato (early 1950s), H. Rollnik (1956), F. Stummel (1956), V. G. Maz'ya (1964), M. Schechter (late 1960s), B. Simon (1970s) and others. In fact, (8) is a modified Stummel-Schechter condition. These are all multiplier-type conditions, but conditions motivated mainly by physical considerations. Condition (2) was also motivated by the Schrödinger operator; however, it appears that the higher-order L_p analogues of (2) were motivated more by purely mathematical considerations, namely by a desire to complete the characterization of multipliers begun by A. Devinatz and I. I. Hirschman (1959–1962) and by R. Strichartz (1967); later contributions included work of J. C. Polking (1972), V. G. Maz'ya-T. O. Shaposhnikova (1979–1981), D. A. Stegenga (1980), and others. The book *Multipliers* follows this second approach and, unfortunately, does not even mention the physical background. (And for that matter, the review [7] ignores the Maz'ya treatment of Schrödinger operators.) Nevertheless the different approaches to multipliers have common roots. Perhaps the appearance of *Multipliers* will spark an interchange of ideas between these two approaches.

3. About the book. I was pleased when I noticed that the Pitman Press was publishing *Multipliers*. Indeed, my own interests over the last ten to fifteen years have been in areas of mathematics quite closely related to work of V. G. Maz'ya—specifically in nonlinear potential theory and its associated theory of L_p-capacities. These are the central concepts used in *Multipliers*. Thus, it is nice to see this body of ideas developed further and applied. Also it is nice to see the work of Maz'ya-Shaposhnikova appear in book form in the Western press. They have many papers on multipliers and related topics scattered throughout the journals. These are now collected, organized and accessible. Furthermore, I have felt for some time that the work of Professor Maz'ya has not received the attention in the West that it deserves. This is especially true with regard to his ideas on multipliers and their relationship to differential equations, e.g., the Schrödinger equation with irregular potentials, as mentioned earlier. This book should help rectify this situation.

The work of Professor Maz'ya, alone or with various colleagues, has appeared now for fully two and a half decades. He has written on many diverse subjects, including the study of Sobolev functions on domains with irregular boundaries (extension theorems, embedding theorems, counterexamples, etc.), and has made extensive contributions to the theory of elliptic boundary value problems, especially in irregular regions. Since many of his original papers are rather difficult to obtain in the West, it is worth knowing that some of this material has been collected in translated editions. With regard to Sobolev-type functions one should consult his work with Yu. D. Burago **[8]** and his three volumes in German in the Teubner-Text zur Mathematik series **[9]**. In fact, it is the bulk of these three volumes that constitutes the new book *Sobolev spaces*, Springer-Verlag, 1985. Almost all of the prerequisites needed to read *Multipliers* can be found in this book. In partial differential equations there is the recent *Elliptic boundary value problems* (Amer. Math. Soc. Transl., vol. 123, 1984) co-authored with B. A. Plamenevskiĭ, N. F. Morozov, and L. Stupyalis, as well as a 1981 Akademie-Verlag manuscript on differential operators in a half-space (in German), co-authored with I. W. Gelman.

Multipliers is, however, a very technical and specialized book that I would recommend mainly as a reference/research text. It contains a wealth of information—results due mainly to the authors—but it is a very exhausting book to read, especially, I suspect, for someone trying to read this material for the first time. The multiplier characterizations are produced with an increasing degree of complexity, from the classical Sobolev spaces on $\mathbf{R}^n$ in Chapter One to Bessel potentials in Chapter Two and then to the Besov space situation (fractional derivatives in L_p using difference quotients) in Chapter Three. Compact multipliers are treated in Chapter Four. It is not until Chapter Six that multipliers for Sobolev spaces on subdomains of $\mathbf{R}^n$ are considered, and then essentially only for Lipschitz domains (only five pages are devoted to the general domain situation). There are several applications of multipliers scattered throughout the first six chapters (including the discussion of the spectrum of a multiplier operator, differentiable maps on surfaces and manifolds, and the implicit function theorem), but it is really Chapter Seven that concentrates solely on applications. Here two problems in the regularity theory of the boundary of a domain in the L_p-theory of elliptic boundary value problems are discussed: the Fredholm property of a differential operator and the solvability of the Dirichlet problem in a Sobolev space on a bounded Lipschitz domain. In this last case, the "boundary maps" are now multipliers on certain Sobolev-type spaces on the boundary. An omission with regard to applications is the failure to mention any connections between the theory of multipliers and the existence of solutions to various nonlinear partial differential equations. One such obvious reference is [**10**]. It is, in fact, of interest to compare the Berger-Schechter conditons for compact multipliers with those of *Multipliers*. There is a very nice and brief summary of the results of the book in the introduction, as well as a large list of references, two indices, and a list of symbols used in the text included at the end. These aid the reader considerably.

4. Nonlinear potential theory. Finally, a few words should be added here about the principal tool used in *Multipliers*, namely nonlinear potential theory. The foundations of this theory were laid down in the late 1960s–early 1970s by several groups and individuals working independently: V. G. Maz'ya and V. P. Havin in Leningrad, L. I. Hedberg in Stockholm, N. G. Meyers and myself in Minneapolis and others including B. Fuglede, J. G. Rešetnjak, J. Serrin, W. Littman, and J. Polking. But even as early as 1961, Professor Maz'ya began to develop his ideas for a capacity based on functions whose derivatives belong to the L_p-spaces rather than the more traditional L_2 or Hilbert space case (Dirichlet space). The basic goal of this theory is to extend the ideas of classical (linear) potential theory (Newtonian potential, capacity, harmonic function, etc.) to a nonlinear setting—in particular, to an L_p-setting, $p \neq 2$. The Dirichlet integral is replaced by $\int |\nabla u|^p\,dx$, $1 < p < \infty$ (or by higher-order analogues), harmonic functions by solutions to the p-Laplace equation $\Delta_p u = 0$, superharmonic functions by $-\Delta_p$ supersolutions, etc. Here the p-Laplacian of u is

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

which becomes the usual Laplacian when $p = 2$. The Riesz decomposition theorem on $\mathbf{R}^n$ implies that superharmonic functions are Newtonian potentials modulo harmonic functions. It turns out that the right version of Newtonian potential in the $p \neq 2$ case is the nonlinear potential

$$\int |x - y|^{1-n}\left(\int |y - z|^{1-n}\,d\mu(z)\right)^{q-1} dy,$$

$1/p + 1/q = 1$, $\mu =$ Borel measure. Notice that these nonlinear potentials are just Newtonian potentials when $p = 2$. However, these new potentials are not $-\Delta_p$ supersolutions. Thus it would seem that the potential theory associated with these nonlinear potentials is at best only very weakly linked to the differential equation side. Nevertheless, there are still very strong parallels to the classical theory including: capacity theory, removable singularity theory, and boundary regularity theory (Wiener criteria) for partial differential equations. Of course, the L_p-methods are much different than the classical ones.

The nonlinear potential theory used in *Multipliers* is pre-1980. In 1983 with the publication of [**11**], T. Wolff removed an inherent difficulty in the theory. One consequence of this theory for *Multipliers* is that certain capacities based on the Besov spaces in Chapter Three need not be introduced, or to say it another way, the capacities based on the Besov spaces of Chapter Three are equivalent to the corresponding capacities for the Bessel potential spaces for all $1 < p < \infty$ (see Proposition 3, p. 115). This would improve the treatment of the fractional-order derivative case in Chapter Three. It is unfortunate that the Hedberg-Wolff results could not have been incorporated into the fabric of *Multipliers*.

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