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THE MODULI SPACE OF A PUNCTURED SURFACE AND PERTURBATIVE SERIES

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0. Introduction. Let F_g^s denote the oriented genus g surface with s punctures, $2g-2+s>0$, $s \geq 1$, and choose a distinguished puncture P of F_g^s. Let $\mathcal{T}_g^s$ be the *Teichmüller space* of conformal classes of complete finite-area metrics on F_g^s (see [**A**]), and let MC_g^s denote the *mapping class group* of orientation-preserving diffeomorphisms of F_g^s (fixing P) modulo isotopy (see [**B**]). When g, s are understood, we omit their mention. In §1 and §2, we report on joint work with D. B. A. Epstein [**EP**] where new and useful coordinates on $\mathcal{T}_g^s$ are given (Theorem 2) and a MC_g^s-equivariant cell decomposition of $\mathcal{T}_g^s$ is described (Theorem 3). There is thus an induced cell decomposition of the quotient $\mathcal{M}_g^s = \mathcal{T}_g^s/MC_g^s$, which is the usual *moduli space* of F_g^s in case $s=1$. In §3, we describe a remarkable connection (see [**P**]) between this cell-decomposition for $s=1$ and a technique from quantum field theory, which allows the computation of certain numerical invariants of $\mathcal{M}_g^s$ (Corollary 6). Analogues of Theorem 3 have been obtained independently by [**BE** and **H**] using different techniques. Furthermore, Corollary 7 is in agreement with some recent work in [**HZ**].

Let M denote Minkowskii 3-space with bilinear pairing $\langle \cdot, \cdot \rangle$ of type $(+,+,-)$, and let $L^+ \subset M$ denote the (open) positive light-cone. The uniformization theorem (see [**A**]) allows us to identify $\mathcal{T}_g^s$ with the space of (conjugacy classes of faithful and discrete) representations of $\pi_1(F_g^s)$ in $SO(2,1)$ (as a Fuchsian group of the first kind in the component of the identity).

I. Coordinates on $\mathcal{T}$. Suppose $\pi_1 F = \Gamma \in \mathcal{T}$, and choose a parabolic transformation $\gamma \in \Gamma$ corresponding to the puncture P. γ fixes a unique ray in L^+, and we choose a point $z \in L^+$ in this ray. If c is a bi-infinite geodesic in F which tends in both directions to P (to be termed simply a *geodesic* in the sequel), let $\gamma(c) \in \Gamma$ denote the corresponding translation, and define the λ-*length* of (the homotopy class of) c to be $\lambda_\Gamma(c) = \sqrt{-\langle z, \gamma(c) z\rangle}$. When Γ is understood, we denote $\lambda(c) = \lambda_\Gamma(c)$. If h is a Γ-horosphere about P and c is a Γ-geodesic, then we define $d_h(c)$ to be the Γ-hyperbolic length along c from h back to h.

LEMMA 1. *If c_1 and c_2 are geodesics, then*

$$\lim_{h \to P} \exp\{d_h(c_1) - d_h(c_2)\} = [\lambda(c_1)/\lambda(c_2)]^2.$$

It follows that λ-lengths are natural in the sense that if $\varphi \in MC$, $\Gamma \in \mathcal{T}$, and c_1, c_2 are geodesics, then $\lambda_{\varphi^*\Gamma}(c_1)/\lambda_{\varphi^*\Gamma}(c_2) = \lambda_\Gamma(\varphi^{-1}c_1)/\lambda_\Gamma(\varphi^{-1}c_2)$,

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where φ^* denotes push-forward of conformal type by φ; moreover, the set of ratios of λ-lengths for a fixed Γ are discrete.

We define an *ideal triangulation* Δ of F to be a decomposition of F by geodesics into regions whose doubles are thrice-punctured spheres.

THEOREM 2. *λ-lengths of edges of Δ give $\mathbf{R}$-analytic projective coordinates on $\mathcal{T}$. Furthermore, MC acts on these coordinates by polynomials.*

For the first part of the theorem, we use the λ-length data to build an ideal tesselation of the hyperbolic plane and apply Poincaré's Theorem to associate a conformal structure on F. For the second part, we claim that the following move on ideal triangulations acts transitively: remove the diagonal e of an ideal quadrilaterial Q of Δ, replacing it by the other diagonal f of Q. (This well-known result follows from our Theorem 3 below.) If (a, c), (b, d) are the pairs of opposite edges of Q, then one computes in M that $\lambda(e)\lambda(f) = \lambda(a)\lambda(c) + \lambda(b)\lambda(d)$. Since λ-lengths are natural by Lemma 1, the theorem follows.

REMARKS. (1) The previous equation is exactly Ptolemy's theorem on side lengths of a Euclidean quadrilateral inscribed in a circle.

(2) The action of MC on λ-lengths is explicit; computer work has been done.

II. The cell decomposition of $\mathcal{T}$. Consider now the orbit Γz of z in L^+. Lemma 1 guarantees that Γz does not accumulate in L^+ (even though the action of Γ on L^+ is ergodic). Furthermore, the extremal edges of the (Euclidean) convex hull of Γz can be shown to project to a collection $\Delta(\Gamma)$ of disjoint geodesics in F, and regions of $F - \Delta(\Gamma)$ are either simply connected or puncture-parallel; we call such a collection of geodesics in F an *ideal cell decomposition* of F. Let $\mathcal{H}$ denote the poset consisting of all

$$C(\Delta) = \{\Gamma \in \mathcal{T} : \Delta(\Gamma) = \Delta\}, \qquad \Delta \text{ an ideal cell decomposition},$$

where $C(\Delta) < C(\Delta')$ if $\Delta \subset \Delta'$.

THEOREM 3. *$\mathcal{H}$ is an MC-equivariant cell decomposition of $\mathcal{T}$. Furthermore, $\mathcal{H}$ extends naturally to a cell decomposition of a natural compactification $\overline{\mathcal{T}}$ of $\mathcal{T}$ so that cells are finite-sided.*

To prove Theorem 3, one first describes $C(\Delta)$ in λ-length coordinates (with respect to Δ) by a system of coupled nonlinear inequalities. Again using naturality of λ-lengths, we see that $\mathcal{H}$ is an MC-invariant decomposition of $\mathcal{T}$, and it remains to check that each $C(\Delta)$ is contractible. We describe a contraction on the set of λ-lengths parametrizing $C(\Delta)$, which respects the system of inequalities; this step is quite delicate and involves some estimates on side lengths of Euclidean polygons. (See Remark 1 above.) The extension to a compactification is accomplished by adjoining cells corresponding to families of geodesics disjointly embedded in F which are not ideal cell decompositions.

The duals of ideal cell decompositions of F_g^s are *certain* spines G in F_g^s, and if $s = 1$, then all spines arise. We will also consider the dual spine Σ of $\mathcal{H}$ on $\mathcal{T}$. Cells of Σ are given by (isotopy classes of certain) pairs (F, G) with partial ordering given by Whitehead collapses of graphs. Σ is contractible, and MC acts on Σ with finite isotropy groups.

REMARK. Suppose $G \subset F$ is a spine of F which is dual to a cell decomposition. There is a $\Gamma_G \in \mathcal{T}$ so that the topological symmetry group of the pair (F, G) acts as a group of conformal symmetries of Γ_G. The corresponding matrix groups are explicitly computable and have interesting diophantine properties.

III. Perturbative series computations. A well-known numerical invariant of the moduli space $\mathcal{M}_g^1$ of F_g^1 is the *virtual* (or *orbifold*) *Euler characteristic* $\chi_g = \chi(Z)/[MC_g^1 : Z]$, where $Z < MC_g^1$ is finite-index torsion-free and $\chi(Z)$ is the usual Euler characteristic. We will compute χ_g along with a collection of further numerical invariants.

We define a *fat graph* G to be a planar projection of a graph in $\mathbf{R}^3$; each vertex of G is required to be at least tri-valent. A neighborhood of the vertex set inherits an orientation from $\mathbf{R}^2$, and we attach orientation-preserving bands along the edges of G to build a pair $(F(G), G)$, where G is a spine of the surface $F(G)$. We define $v_k(G) = \#\{k\text{-valent vertices of } G\}$, $\lambda(G) = \#\{\text{boundary components of } F(G)\}$, and let $\Gamma(G)$ ($[G]$, resp.) denote the automorphism group (isomorphism class, resp.) of the oriented pair $(F(G), G)$.

In what follows, we compute

$$\phi(I, N) = \sum_{[G] \text{ with } -2\chi(G)=I} \frac{(-1)^{\Sigma v_k(G)} N^{\lambda(G)}}{\#\Gamma(G)}.$$

The argument I determines the Euler characteristic of contributing fat graphs, and for each I, ϕ is a polynomial in N. As a special case, the end of §2 guarantees that χ_g is the coefficient of N in $\phi(4g-2, N)$ which can be taken as motivation for our interest in ϕ.

THEOREM 4. *We have the following equality:*[1]

$$\phi(I, N) = \sum_{\substack{\text{tuples } v_k \text{ with} \\ \Sigma(k-2)v_k = I \\ \text{so that } v_1 = v_2 = 0}} \frac{(-1)^{\Sigma v_k}}{2^{N/2}\pi^{N^2/2}\prod_{k\geq 1} v_k!}$$

$$\cdot \int_{M\in H^N} \prod_{k\geq 1} \left[\frac{\operatorname{trace} M^k}{k}\right]^{v_k} \exp\left(-\frac{1}{2}\operatorname{trace} M^2\right) dM,$$

where dM is the unitary-invariant product of Lebesgue measures

$$dM = \left(\prod_{i=1}^N dM_{ii}\right) \prod_{1\leq i<j\leq N} d(\operatorname{Re} M_{ij})\, d(\operatorname{Im} M_{ij})$$

on the $N \times N$ Hermitians H^N.

The technique of proof is an extension of one known amongst physicists as Feynman diagrams/perturbative series (see [**BIZ**]) and will not be taken up

[1] NOTE ADDED IN PROOF. The right-hand side of this equality bears a striking resemblance to Weil's formula for the total Chern class of a bundle.

here. It is remarkable that this technique so effectively captures the combinatorics of our complex Σ, at least when $s = 1$.

THEOREM 5. *We have the following equality.*

$$\phi(2I, N) = \frac{(-1)^I}{I!} \frac{\partial^I}{\partial t^I}\bigg|_{t\downarrow 0} \left[\frac{\sqrt{2\pi t}(et)^{-t^{-1}}}{\Gamma(t^{-1})}\right]^N \prod_{p=1}^{N-1} (1-pt)^{N-p}.$$

To prove this result, one passes from the integral in Theorem 4 to an integral over $\mathbf{R}^N$. In so doing, one introduces the factor $\prod_{1\le i<j\le N}(x_i - x_j)^2$ in the integrand. The integral of such a factor against $\prod_{i=1,N} d\mu(x_i)$ for some measure $d\mu$ on $\mathbf{R}$ can be expressed in terms of the zeroth moment of $d\mu$ and the coefficients of the recursion relation for the orthogonal polynomials of $d\mu$. We construct a generating function $\hat{\phi}(t, N)$ for the quantities $\phi(I, N)$ so that the resulting integral is of this form for some measure $d\mu_t$ whose orthogonal polynomials and moments are explicitly computable. Unfortunately, there are convergence problems for $t = 0$, and we must truncate the integrals spatially and take limits to pursue this program. Thus, $\{\phi(2I, N): I \ge 0\}$ arises as the set of coefficients in the asymptotic series at zero of the function in Theorem 5.

We define $\theta(I, N)$ similarly to $\phi(I, N)$ but summing only over *connected* fat graphs. Taking logarithmic derivatives and using a variant of Stirling's formula gives our main result on fat graphs.

COROLLARY 6. *We have the following equality.*

$$\theta(2I, N) = (-1)^I \left[\frac{-N^{I+2}}{I(I+1)(I+2)} + \sum_{k=1}^{[(I+1)/2]} \binom{I-1}{2k-2} \frac{B_{2k}}{2k} \frac{N^{I+2-2k}}{I+2-2k}\right],$$

where B_{2k} is the $2k$th Bernoulli number and $[\cdot]$ denotes integral part.

Since a fat graph with $\lambda = 1$ is necessarily connected, we find

COROLLARY 7. *The virtual Euler characteristic of moduli space is*

$$\chi_g = \textit{coefficient of } N \textit{ in } (\theta(4g-2, N)) = \frac{-B_{2g}}{2g} = \varsigma(1-2g),$$

where ς is the Riemann zeta function.

Only a small part of the data about fat graphs obtained in Corollary 6 is used in Corollary 7. The remaining information is likely to be related to the action of MC_g^s on our decomposition of $\mathcal{T}_g^s$.

REFERENCES

[**A**] W. Abikoff, *The real-analytic theory of Teichmüller space*, Lecture Notes in Math., vol. 820, Springer-Verlag, Berlin and New York, 1980.

[**B**] J. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Studies, no. 82, 1974.

[**BE**] B. Bowditch and D. B. A. Epstein, *Triangulations associated with punctured surfaces*, preprint (1985).

[**BIZ**] D. Bessis, C. Itzykson, and J. B. Zuber, *Quantum field theory techniques in graphical enumeration*, Adv. Appl. Math. **1** (1980), 109–157.

[**EP**] D. B. A. Epstein and R. C. Penner, *Teichmüller spaces of punctured surfaces*, preprint (1984).

[**H**] J. Harer, *The virtual cohomological dimension of the mapping class group of an oriented surface*, Invent. Math. (1986).

[**HZ**] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, preprint (1985).

[**P**] R. C. Penner, *Perturbative series and moduli space*, preprint (1985).

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