BOOK REVIEWS

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Sequences and series in Banach spaces, by Joseph Diestel, Graduate Texts in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1984, xi + 261 pp., $38.00. ISBN 0-387-90859-5

Many of the problems that motivated the development of functional analysis had to do with sequences and series in various linear spaces. As outstanding examples, consider the expansion of a "general" function in terms of a given sequence of functions (Fourier series or, more generally, the expansion with respect to eigenfunctions of integral or differential operators), or the different convergence notions of sequences of measurable functions (almost everywhere, in measure, in L^p for some $1 \leqslant p \leqslant \infty$), or the convergence of certain sequences of operators (e.g., convolutions with suitable approximate identities).

The results sought were either of a general nature—i.e., true in all Banach spaces (or other families of linear topological spaces)—or results in particular spaces: e.g., Hilbert spaces, L^p spaces, reflexive spaces, etc. The uniform boundedness principle is a typical result of the first type. Results of the second type occur in almost every instance where functional analysis is applied to a specific problem. For example, when dealing with a differential equation, one usually considers a Hilbert space (or L^p, or appropriate Sobolev space) setting and uses the special properties of this space to solve the equation. It is then a different problem (in which, again, the structure of the space usually plays an important role) to prove that the solution found is, in fact, smooth.

With time, linear topological spaces, and Banach spaces in particular, became the subject of an independent study. A beautiful and deep theory emerged, involving the analysis of the structure and classification of general Banach spaces as well as the detailed study of specific Banach spaces, many of which are the common spaces of analysis.

It is not at all surprising that many of the invariants developed for this study had to do with the behavior of sequences and series in the various spaces. Here is a quick early example. A sequence (x_n) in a Banach space is called weak-Cauchy if $\lim x^*(x_n)$ exists for every continuous linear functional x^*. A Banach space is called weakly sequentially complete if every weak-Cauchy sequence in it converges weakly. It is already shown in Banach's *Théorie des opérations linéaires* that $L^1[0,1]$ is weakly sequentially complete, while $C[0,1]$ is not. Thus, one cannot embed $C[0,1]$ as a closed subspace of $L^1[0,1]$.

This is a very simple, yet typical, example. It involves relations between a space and its dual and convergence with respect to one of the natural topologies that a Banach space carries. These "topological" ideas were the main subject of the early development of the theory, and many deep and important results appeared. To mention just a few of these early results, recall Mazur's Theorem: *If* (x_n) *converges weakly to* x, *some convex combinations of the* x_n*'s converge to* x *in norm*, or the Orlicz-Pettis Theorem: *If* $\Sigma\, x_{n_i}$ *converges weakly for all subsequences of* (x_n), *then these series are, in fact, norm convergent*, or the Eberlein-Smulian Theorem: *A set is weakly compact iff it is weakly sequentially compact.*

As the theory of Banach spaces developed, it became clear that their structure could be very complicated. A dramatic example was given in 1972 by Enflo's construction of a Banach space without a basis, and in fact without the approximation property—a much weaker "regularity" condition. In an attempt to develop some theory for general spaces, much effort was put into trying to find, for any Banach space X, at least a "regular" subspace of X. For example, it was already proved by Mazur (and mentioned in Banach's book) that every infinite-dimensional Banach space contains a basic sequence: i.e., a basis for its closed linear span. Can one say more? Could it be true that, in fact, every space must contain a basic sequence with some additional properties? For a long time the conjecture was that every space should contain a subspace isomorphic to one of the standard sequence spaces—c_0 or l_p for some $1 \leqslant p < \infty$. A counterexample was constructed in 1974 by B. S. Tsirelson. (A detailed exposition of this important space and its modifications will appear soon in a book by P. Casazza and T. Shura.)

Because one cannot hope to get c_0 or l_p in every Banach space, one should settle for less. Could it be true that every infinite-dimensional Banach space contains an unconditional basic sequence? (Recall that a basis (x_n) for a Banach space X is called unconditional if for every $x \in X$ the expansion $x = \Sigma\, a_n x_n$ converges to x under any rearrangement of its terms. Equivalently, $\Sigma \pm a_n x_n$ converges for all choices of signs.) This is one of the major open problems in the theory. An example of Maurey and Rosenthal shows that the naive hope that every basic sequence contains an unconditional subsequence fails.

But not all results in this direction are counterexamples. There have been some very significant positive results. After the pioneering work of H. P. Rosenthal, he and others gave a series of necessary and sufficient conditions, all relatively simple to verify, for a space to contain a subspace isomorphic to l_1. The ideas involved here are of classical flavor, and the conditions have to do with topological structures and the action of linear functionals. The main result in the first Rosenthal paper goes as follows: Given a sequence (x_n) in a Banach space, either it has a weak-Cauchy subsequence, or else it has a subsequence equivalent to the unit vector basis of l_1.

Another success was obtained by B. Maurey and J. L. Krivine. Following a breakthrough by D. Aldous in the understanding of the structure of subspaces of $L^1[0,1]$, they identified a large class of spaces that necessarily contain l_p for some $1 \leqslant p < \infty$. The situation here is, however, much less clear than the l_1 case. The condition here is only sufficient. It is an isometric condition that can be destroyed by renorming and is usually quite hard to check.

All these results are more or less a direct continuation of the study of Banach spaces as it was originally initiated. The main objects of study are infinite-dimensional spaces, and the properties studied involve topological and infinite-combinatorial considerations. Starting from the mid-1960s there has been a major change of direction—more and more emphasis has been placed on the "local theory"—i.e., the theory of finite-dimensional spaces and the ways they can be "put together" to form an infinite-dimensional one. The 1968 paper of J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in* $\mathcal{L}_p$ *spaces and their applications*, which was based on A. Grothendieck's work in the early 1950s, was most influential in this trend, although many results appeared much earlier.

Many of the invariants of the infinite-dimensional theory have no finite-dimensional analogue. This is true, for example, for those invariants that relate the behavior of sequences in the various natural topologies of an infinite-dimensional space. Other invariants could be modified to apply to finite-dimensional spaces. For example, in finite-dimensional spaces a series Σx_n converges unconditionally (i.e., $\Sigma \pm x_n$ converges for all choices of signs) iff it converges absolutely (i.e., $\Sigma\|x_n\| < \infty$). The famous Dvoretzky-Rogers Theorem says that this characterizes finite-dimensional spaces. It turns out, however, that one can still compare unconditional and absolute convergence even in finite-dimensional spaces. A routine application of the closed graph theorem shows that for a given finite-dimensional space X, there is a constant K so that $\Sigma\|x_n\| \leqslant K \sup\|\Sigma\pm x_n\|$ for all unconditionally convergent series Σx_n in X (where the "sup" is taken over all choices of signs). A typical problem of local theory is to estimate this K and to analyze its dependence on the space X. Thus the smaller K is, the "more similar" are absolute and unconditional convergences in X. In fact, this was exactly the approach of the original Dvoretzky-Rogers proof. They proved that if $\dim X = d$, then necessarily $K \geqslant d^{1/4}$ (it is now known that $K \geqslant d^{1/2}$). It was then a simple matter to produce in an infinite-dimensional space a sequence of finite series with increasingly worse ratios between $\Sigma\|x_n\|$ and $\sup\|\Sigma\pm x_n\|$ and to put these together so as to give an unconditionally convergent series for which $\Sigma\|x_n\|$ is infinite.

This example is very typical. The local theory is necessarily quantitative. Qualitatively all spaces of a given finite dimension are the same. Thus the invariants are quantitative, and usually one studies their asymptotic behavior as the dimension increases.

The local theory has seen enormous success in the last decade. Using tools from diverse fields, such as integral geometry, combinatorics, probability, and harmonic analysis, many delicate and precise results have been obtained. The forthcoming Lecture Notes of V. D. Milman and G. Schechtman and G. Pisier's forthcoming CBMS notes, based on his Missouri lectures, should provide a very good idea of this theory, its objectives and techniques, as well as its relations with other branches of mathematics.

Let me now discuss the book under review. It covers much of the classical theory as well as many recent developments. The emphasis is on the qualitative, topological and infinite-combinatorial aspects of the infinite-dimensional theory.

Assuming only basic functional analysis and measure theory, the author gives a thorough, complete, and self-contained presentation of many of the fundamental results, including weak- and weak*-topologies, the Eberlein-Smulian Theorem, the Orlicz-Pettis Theorem, basic sequences, the Dvoretzky-Rogers Theorem (via p-absolutely summing operators), weak- and weak*-convergence in classical spaces, Choquet theory, Ramsey's theory, and much more.

He then goes on to present Rosenthal's l_1 theorem and many of its variations, including the important Josefson-Nissenzweig Theorem, which states that every infinite-dimensional dual space contains a w*-convergent sequence with no norm-convergent subsequence. This sets the stage for a thorough discussion of spaces X so that the unit ball of X^* is w*-sequentially compact.

This short list is very far from an exhaustive description of the contents of the book, which contains many more, important results. The book is written in textbook style, and the author has been very careful to present each topic in great detail and to present some of the very recent developments on almost every topic. Each chapter has a set of exercises that further develop the theory and a section of Notes and Remarks with important additions to, and ramifications of, the material presented in the text.

The selection of topics reflects the author's personal interests very much. While one can argue about the inclusion or omission of certain specific topics (e.g., the Krivine-Maurey Theorem is very much in the spirit of the book and I would have included it), it is obvious that some selection must be made. The only regret I have in this respect is the neglect of the quantitative theory. The author apologizes in the introduction for not including the theory of type and cotype, as it would have greatly increased the length of the book. But much less than a thorough presentation would have sufficed. A little more emphasis on the quantitative aspects of topics presented, such as p-absolutely summing operators, finite versions of the Dvoretzky-Rogers Theorem, or a few more remarks and exercises showing finite-dimensional analogues of topics discussed, together with few selected results from local theory would have changed the balance, and made the book more representative of current interests in research.

The book has surprisingly many misprints for this prestigious series, but these are not more than slightly annoying, and I don't expect them to present difficulties even for the novice.

With the rapid advance of Banach space theory, it becomes increasingly difficult to enter the field. There is a huge gap between the introductory functional analysis texts and the research-level treatises, and very little intermediate or textbook level material is available. Indeed, most of the material in this book appears for the first time in textbook-level form. Within the framework that the author set for himself, namely, an intermediate-level exposition of qualitative infinite-dimensional theory, he has done a very thorough job.

The book is a most welcome addition to the unfortunately small number of books in the field.

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