BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 13, Number 1, July 1985

DEHN SURGERY ON KNOTS

BY MARC CULLER, C. McA. GORDON, J. LUECKE AND PETER B. SHALEN

Let M be a compact, connected, orientable, irreducible 3-manifold such that ∂M is a torus. An isotopy class c of unoriented simple closed curves in ∂M will be called a *slope*. A closed 3-manifold $M(c)$ may be constructed by attaching a solid torus J to M so that c bounds a disk in J.

If c and d are two slopes, we denote their (minimal) geometric intersection number by $\Delta(c, d)$.

THEOREM. *Suppose that M is not a Seifert fibered space. If $\pi_1(M(c))$ and $\pi_1(M(d))$ are cyclic, then $\Delta(c, d) \leqslant 1$. In particular, there are at most three slopes c such that $\pi_1(M(c))$ is cyclic.*

This result is sharp; Fintushel–Stern and Berge have given examples of hyperbolic knots in S^3 for which two Dehn surgeries give lens spaces.

In the statements of the following corollaries we use rational numbers as in **[R]** to parametrize the nontrivial Dehn surgeries on a knot K in S^3. We will denote by $K(r)$ the result of r-surgery on K.

COROLLARY 1. *If K is not a torus knot and $r \in \mathbf{Q}$, then $\pi_1(K(r))$ can be cyclic only if r is an integer. Moreover, there are at most two such integers r, and if there are two then they must be successive.*

COROLLARY 2. *If K is a nontrivial knot and $r \in \mathbf{Q}$ is not equal to* 1 *or* -1 *then $K(r)$ is not simply-connected. Moreover, $K(1)$ and $K(-1)$ cannot both be simply-connected.*

COROLLARY 3. *Up to unoriented equivalence, there are at most two knots whose complements are of a given topological type.*

COROLLARY 4. *If K is a nontrivial amphicheiral knot and $r \in \mathbf{Q} - \{0\}$, then $\pi_1(K(r))$ is not cyclic. In particular, K has Property P.*

COROLLARY 5. *Knots of Arf invariant* 1 *are determined up to unoriented equivalence by their complements.*

Whitten **[W]**, using work of Johannson **[Jo1]**, shows that Corollary 1 implies the following result.

COROLLARY 6. *Prime knots with isomorphic groups have homeomorphic complements.*

Received by the editors February 7, 1985.
1980 *Mathematics Subject Classification*. Primary 57M25, 57R65.

The theorem states that, with a very small set of exceptions, the group $\pi_1(M(c))$ is not cyclic. We prove this by showing that either

(*) there exists an incompressible surface in $M(c)$; or

(**) there exists a representation of $\pi_1(M(c))$ into $PSL_2(\mathbf{C})$ with noncyclic image.

The proof reduces to the case where M is atoroidal, using part (a) of

PROPOSITION 1. (a) *If M contains an incompressible nonperipheral torus which compresses in $M(c)$ and $M(d)$, then either $\Delta(c, d) \leqslant 1$ or $(M, \partial M)$ is cabled in the sense of* [**GL**].

(b) *Suppose that* $\dim H_1(M; \mathbf{Q}) > 1$. *If $M(c)$ and $M(d)$ have cyclic first homology groups and are not Haken manifolds then $\Delta(c, d) \leqslant 1$.*

The proof of Proposition 1 is a combinatorial analysis of the intersection of the two planar surfaces in M corresponding, in (a), to the compressing disks for the torus in $M(c)$ and $M(d)$, and, in (b), to the nonseparating 2-spheres in $M(c)$ and $M(d)$ which would exist if $M(c)$ and $M(d)$ were not Haken.

We define a slope c to be a *boundary slope* if M contains an incompressible nonperipheral surface F with $\partial F \neq \varnothing$ such that each component of ∂F has slope c. The next proposition, together with Proposition 1 (b), establishes the inequality in the conclusion of the theorem when one of the slopes is a boundary slope.

PROPOSITION 2. *If c is a boundary slope and* $\dim H_1(M; \mathbf{Q}) = 1$, *then either*

(i) *$M(c)$ is a Haken manifold; or*

(ii) *$M(c)$ is a connected sum of two (nontrivial) lens spaces; or*

(iii) *M contains a closed incompressible surface which remains incompressible in $M(d)$ whenever $\Delta(c, d) > 1$.*

For the proof of Proposition 2, let F be an incompressible nonperipheral surface in M with $\partial F \neq \varnothing$ having boundary slope c and with the minimal number of boundary components. Consider the manifold X obtained by cutting M along F. If there are enough compressing disks for ∂X in X, one shows either that the capped-off surface $\hat{F}$ in $M(c)$ is an incompressible surface of positive genus and (i) holds, or that $\hat{F}$ is an essential 2-sphere and (ii) holds. (The proof uses an extension of a result of Jaco [**Ja**] and Johannson [**Jo2**] giving conditions under which the addition of a 2-handle to a 3-manifold will yield a boundary-irreducible manifold.) Otherwise X, and hence M, contains a closed incompressible surface which is shown, by a refinement of the combinatorial analysis used in the proof of Proposition 1, to remain incompressible in $M(d)$ if $\Delta(c, d) > 1$. This gives conclusion (iii).

Finally, we consider the case that M is atoroidal and that c and d are nonboundary slopes. Thurston's Geometrization Theorem implies that the interior of M has a hyperbolic structure of finite volume. We define, as in [**CS**], a complex affine curve X in the space of characters of representations of $\pi_1(M)$ in $SL_2(\mathbf{C})$. We identify $L = H_1(\partial M; \mathbf{Z})$ with a lattice in the vector $V = H_1(\partial M; \mathbf{R})$. Let $e\colon L \to \pi_1(M)$ denote the inverse of the Hurewicz isomorphism followed by the inclusion $\pi_1(\partial M) \to \pi_1(M)$. Each $\gamma \in L$ defines a regular function $I_\gamma\colon X \to \mathbf{C}$ by $I_\gamma(\chi) = \chi(e(\gamma))$ (cf. [**CS**]). One shows that there is a piecewise linear norm $\|\ \|$ on V such that for each γ in the lattice $L \subset V$, degree $I_\gamma = \|\gamma\|$. Then the ball of radius $m = \min_{0 \neq \gamma \in L}(\|\gamma\|)$ is a convex polygon B such that $B = -B$, and the interior of B contains no points of L. One concludes that, in terms of the natural area element on V, B has area at most 4.

We shall identify a slope with a pair $\{\pm\gamma\}$ of primitive elements of L. The following result is proved by the techniques of **[CS]**.

PROPOSITION 3. (a) *Each vertex of B is a rational multiple of $\gamma \in L$, where $\{\pm\gamma\}$ is a boundary slope.*

(b) *If c is a nonboundary slope then either $c = \{\pm\gamma\}$, with $\gamma \in B$, or else one of the conclusions* $(*)$ *or* $(**)$ *holds for $M(c)$.*

Suppose now that $c = \{\pm\gamma\}$ and $d = \{\pm\delta\}$ are nonboundary slopes and that $M(c)$ and $M(d)$ satisfy neither $(*)$ nor $(**)$. Consider the parallelogram $P \subset V$ with vertices $\pm\gamma$ and $\pm\delta$. By (a) we have

$$\Delta(c, d) = \tfrac{1}{2}\text{Area } P \leqslant \tfrac{1}{2}\text{Area } B \leqslant 2$$

and equality would imply that γ and δ are vertices of B. By (b) we would have that c and d were boundary slopes, a contradiction. This completes the proof of the theorem.

REFERENCES

[CS] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of* 3-*manifolds*, Ann. of Math. (2) **117** (1983), 109–146.

[GL] C. McA. Gordon and R. A. Litherland, *Incompressible planar surfaces in* 3-*manifolds*, Topology Appl. **18** (1984), 121–144.

[Ja] W. Jaco, *Adding a* 2-*handle to a* 3-*manifold*: *An application to Property R*, Proc. Amer. Math. Soc. **92** (1984), 288–292.

[Jo1] K. Johannson, *Homotopy equivalences of* 3-*manifolds with boundaries*, Lecture Notes in Math., vol. 761, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[Jo2] ______, *On surfaces in one-relator* 3-*manifolds*, preprint.

[R] D. Rolfsen, *Knots and links*, Publish or Perish, Inc., 1979.

[W] W. Whitten, *Knot complements and groups*, preprint.

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 2223 FULTON STREET, BERKELEY, CALIFORNIA 94720