Why should one study (11) at all? If this is answered satisfactorily (and the reviewer believes it might), why should one adopt (13) as a definition of solution (especially because it leads to discrepancies)?

This book, and part of the literature on impulsive ODE, are fundamentally flawed.

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Einhüllende Algebren halbeinfacher Lie-Algebren, by Jens C. Jantzen, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 3, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1983, 298 pp., DM 118; Approx. U.S. $45.80. ISBN 3-5401-2178-1

One of the fundamental problems in abstract harmonic analysis is the determination of the set of (equivalence classes of) irreducible unitary representations of a topological group G. These are continuous homomorphisms of G into the group of unitary operators on a Hilbert space; one assumes, in addition, that the Hilbert space has no nontrivial closed subspaces invariant under the whole group. This is a nonlinear problem, in the sense that group elements and unitary operators can be multiplied, but not added. It is tempting to look for ways to linearize things, for example because of the great success that idea enjoys in the elementary representation theory of finite groups. (There one considers the convolution algebra of all functions on the group. The representation theory of that algebra and of the group are essentially the same.) If G is a locally compact group, there is something called $C^*(G)$, which is a completion of the convolution algebra of L^1 functions on G in an appropriate sense. Any unitary representation π of G gives rise to a representation of $C^*(G)$—that is, to a homomorphism of $C^*(G)$ into the algebra of bounded operators on the Hilbert space. Write I_π for the kernel of this homomorphism; it is an ideal in $C^*(G)$. I_π is called *primitive* if π is irreducible. These ideals play a significant part in abstract harmonic analysis. Two interesting facts appear in that development. First, the primitive ideal I_π often determines the irreducible unitary representation π uniquely. Second, when this uniqueness fails, the ideals are in some respects more important than the representations.

Despite the successes of these ideas, the C^* algebra remains a rather abstract and complicated object, which does not lend itself well to some kinds of calculations. For simple Lie groups, for example, much of the detailed information that is available about the C^*-algebra is based on *a priori* knowledge of the representation theory. One could still use some simpler algebra with which to work. For a connected Lie group G, a natural candidate is the universal enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g}$ of G. (Because we consider only complex vector spaces, a real algebra and its complexification have the same representations.) The enveloping algebra has a relatively simple structure about which a great deal is known. For example, the Poincaré-Birkhoff-Witt theorem provides a basis for the algebra as a complex vector space. The penalty for using it is that the connection between representations of G and representations of $U(\mathfrak{g})$ is not nearly so good as for the C^*-algebra. Nevertheless, by 1953, Harish-Chandra had shown that for a semisimple group G, there is a bijection between the set of irreducible unitary representations of G, and a certain class of irreducible modules for $U(\mathfrak{g})$ **[10]**. The stage appeared to be set for a complete understanding of unitary representations by algebraic methods.

Fifteen years later the picture had not changed very much. Harish-Chandra had made enormous progress on a variety of important problems in harmonic analysis, but a detailed understanding of all irreducible unitary representations seemed less attainable then than it had in the early 1950s. At about this time Dixmier made a suggestion, borrowed perhaps partly from the operator algebra ideas mentioned earlier and partly from abstract algebra. Since the irreducible modules for $U(\mathfrak{g})$ appear to be too complicated to understand and classify directly, one should study only their annihilators. The annihilator of an irreducible module for a ring is by definition a *primitive ideal* in the ring. He was therefore focussing attention on $\operatorname{Prim} U(\mathfrak{g})$, the set of all primitive ideals in $U(\mathfrak{g})$. Examples indicated that this was a set one might hope to understand in considerable detail.

Soon Dixmier had formulated a precise conjecture describing $\operatorname{Prim} U(\mathfrak{g})$ when $\mathfrak{g}$ is a solvable Lie algebra. If $\mathfrak{g}$ is the Lie algebra of a connected (solvable) algebraic group G, it says that $\operatorname{Prim} U(\mathfrak{g})$ is in natural bijection with the orbits of G on the maximal ideals in $S(\mathfrak{g})$. This conjecture was proved through the efforts of a number of people in the early 1970s; an account may be found in [**6** or **8**]. The irreducible unitary representations of solvable Lie groups were already fairly well understood, however [**2**], so the result was of interest more as pure algebra than as a tool for representation theory.

The next natural case to consider was that of a semisimple Lie algebra $\mathfrak{g}$. (From this point on, incidentally, I will be describing the developments reported in Jantzen's book.) These algebras do not lend themselves to reduction arguments using ideals in the Lie algebra, which were critical to the solvable case. The fundamental result which gets the subject off the ground is Duflo's theorem [**9**]: every primitive ideal in the enveloping algebra of a semisimple Lie algebra is the annihilator of a very special kind of irreducible module (a highest-weight module). Duflo had pushed Dixmier's idea one step further. One studies general irreducible modules by studying their primitive ideals; and one studies primitive ideals by studying nice irreducible modules of which they are the annihilators.

Highest-weight modules for semisimple Lie algebras were introduced by Harish-Chandra more than thirty years ago for two specific problems (the constructions of finite-dimensional representations and of certain discrete series). After Harish-Chandra solved the problems, they were essentially abandoned until the thesis of Verma [**15**]. Bernstein, Gelfand, and Gelfand, and later Jantzen, developed a beautiful and powerful theory from these beginnings [**4, 12**]. A key idea of Bernstein, Gelfand, and Gelfand was the use of tensor products of highest-weight modules and finite-dimensional representations. Using this idea, Jantzen showed that every highest-weight module is part of a "coherent family" of such modules, parametrized roughly by the finite-dimensional irreducible representations of $\mathfrak{g}$. (The finite-dimensional representations themselves constitute such a family.) This is Jantzen's *translation principle*, which in one form or another is at the heart of much of what has been done about semisimple Lie groups over the past ten years.

Just as the theory of highest-weight modules had reached this level, Duflo proved his theorem that all primitive ideals arise as annihilators of irreducible highest-weight modules. Immediately Borho and Jantzen in [**7**] deduced an enormous range of basic structural results on primitive ideals. The central one was a translation principle: every primitive ideal occurs in a coherent family of primitive ideals, again parametrized roughly by finite-dimensional irreducible representations of $\mathfrak{g}$.

From this point on, much of the work in the field (mostly by Joseph) could be viewed as refining and extending the translation principle. The progress that was made is not very easy to explain until we come to its culmination in the papers [**14**] of Joseph. There the idea is this. Let $\{I_\lambda | \lambda \in \Lambda\} = \mathscr{F}$ be a coherent family of primitive ideals of the sort described above. The parametrizing set Λ is contained in the dual $\mathfrak{h}^*$ of a Cartan subalgebra of $\mathfrak{g}$ and is Zariski dense there. Define a function on Λ by

$$p_{\mathscr{F}}(\lambda) = \text{Goldie rank of } U(\mathfrak{g})/I_\lambda.$$

Here Goldie rank is an invariant of prime Noetherian rings, which may (though it should not) be taken to be the largest possible order of a nilpotent element. Joseph proved that $p_{\mathscr{F}}$ extends to a polynomial function on all of $\mathfrak{h}^*$.

Furthermore, he proved that a primitive ideal is completely determined by two invariants: its intersection with the center of the enveloping algebra, and the polynomial $p_{\mathscr{F}}$ attached to its coherent family.

This theorem of Joseph (which actually says much more about the possibilities for $p_{\mathscr{F}}$ and related matters) is the centerpiece of Jantzen's book. Beginning at approximately the point where Humphreys' book [**11**] ends, he develops all the ideas needed in a very self-contained way. Most of this material is of great interest in its own right, notably the theory of Harish-Chandra modules for complex semisimple groups. (Group representers would probably want this book for those chapters alone.) One can also find out about the Kazhdan-Lusztig conjecture, special Weyl group representations, Gelfand-Kirillov dimension, associated varieties, and many things of a more specialized nature. Proofs are, in general, the best ones available (which means that they are often better than those in the original papers), and the exposition is uniformly good (which means that it is often infinitely better than that in the original papers). There are a few slips of the typesetter or of the mind—I found one every five or six pages when I was reading carefully—but nothing of an especially confusing nature. The convention used for the τ-invariant is natural, convenient, and opposite to the usual one; but I intend that more as a caveat than as a criticism. In general, the book is as readable as Dixmier's [**8**] and should interest a roughly similar audience.

Jantzen has done an admirable job of conveying the ring-theoretic aspects of primitive ideal theory in semisimple enveloping algebras. Nevertheless, it is worth observing that there is another way of looking at the subject which seeks to understand phenomena in more geometric ways. This perspective is nicely summarized in Borho's lectures [**5**], which could serve as a complement to Jantzen's book. (There one can also find a discussion of many open problems.) One should also be aware of the work of Moeglin and Rentschler on primitive ideal theory in the enveloping algebra of a general finite-dimensional Lie algebra. One of the main points there is to reduce matters to the semisimple case, but this is a very complicated project in practice.

With an excellent knowledge of annihilators in hand, it is natural to ask whether we can now say anything new about group representations. A look into the proofs in Jantzen's book is discouraging in this regard: the great advances in primitive ideal theory have relied heavily on progress in representation theory. There is hope, however. Jantzen uses only the representation theory of complex semisimple groups, but his results apply also to annihilators of representations of real groups. Since complex groups are far less complex than real groups, this is a significant advantage. Real theorems about real groups have been proved in this way, but most of them are too complex to explain here.

It seems likely now that the best is yet to come in the primitive ideal/unitary representation connection. Two of the major questions left open by Jantzen's book (computation of the polynomials $p_{\mathscr{F}}$ and of the Goldie fields for primitive quotients) hinge on finding a finite number of particularly interesting primitive ideals attached in some sense to the nilpotent conjugacy classes in $\mathfrak{g}$ (cf. [**14, 3**]). Similarly, representation theory lacks (among other things) a finite set of irreducible unitary representations for each real semisimple G that are attached to the nilpotent G orbits in Lie(G). Experimental evidence indicates that these problems are inextricably connected. Miraculously, their resolution should have implications for automorphic forms (cf. [**1**]). Jantzen has found a subject perfectly suited for an advanced text: one which has reached not the top of the mountain, but a solid ledge with a beautiful view.

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Complex analysis, a functional analysis approach, by D. H. Luecking and L. A. Rubel, Universitext, Springer-Verlag, New York, 1984, vii + 176 pp., $16.00. ISBN 0-3879-0993-1

1. Function theory in functional analysis. Many branches of mathematics owe a debt to classical function theory. This is especially true of functional analysis. Here the archetypal application, due to M. H. Stone (in his famous 1932 book