BMO ESTIMATES AND RADIAL GROWTH
OF BLOCH FUNCTIONS

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1. Introduction. A real harmonic function $u(z)$ in the open unit disk $\mathbf{D}$ belongs to the (real) Bloch space B if

$$(1) \qquad \|u\|_B = \sup\{(1 - |z|^2)|\operatorname{grad} u| : z \in \mathbf{D}\} < \infty.$$

The importance of the class B in both complex and harmonic analysis is highlighted by the following facts:

(a) If $f(z)$ is a univalent analytic function in $\mathbf{D}$, then $u(z) = \log|f'(z)| \in B$ and $\|u\|_B \le 8$; conversely, if $u(z) \in B$ and $\|u\|_B \le 1$, then there is a univalent analytic function $f(z)$ in D such that $u(z) = \log|f'(z)|$ (see [5, pp. 32, 172]).

(b) The functions $u \in B$ coincide (up to a constant) with the derivatives $(\partial/\partial x)U(re^{ix})$ of harmonic functions $U(z)$ such that $U(e^{ix})$ belongs to the Zygmund class Λ_* (see [6, Chapter 7, §3]).

Integrating (1) yields the following "trivial" estimate (we assume $u(0) = 0$):

$$(2) \qquad |u(z)| \le \frac{1}{2}\|u\|_B \log\frac{1 + |z|}{1 - |z|} \qquad (z \in \mathbf{D}).$$

However, for L_p norms of $u(z)$ on circles $|z| = r$ ($0 < r < 1$) a much deeper estimate, due to J. G. Clunie and T. H. MacGregor [1], holds ($p > 0$):

$$(3) \quad \|u(r,\cdot)\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi}|u(re^{ix})|^p\,dx\right)^{1/p} \le A_p\|u\|_B\sqrt{|\log(1 - r)|},$$

where A_p are some constants (the case $p = 2$ was proved earlier by T. M. Flett [2, p. 71]). As shown in [1], (3) implies that

$$(4) \qquad \lim_{r\to1^-}\frac{u(re^{ix})}{|\log(1 - r)|^\gamma} = 0$$

for almost all x whenever $\gamma > 1/2$.

The need to better understand the nature of such "nontrivial" estimates as (3) and (4) is the main motivation of this paper. In this context the use of the BMO norm for estimating Bloch functions on the circles $|z| = r$ looks more promising, since both norms are invariant under Möbius shifts of the argument.

In what follows, a simple but sharp BMO estimate for Bloch functions is proved. This estimate, together with the John-Nirenberg theorem [3], is then used to obtain a stronger form of the asymptotic estimate (4).

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2. BMO estimate.

DEFINITION 1. B is the real Banach space of (real) harmonic functions $u(z)$ in $\mathbf{D}$ modulo constants with norm (1).

DEFINITION 2. BMO is the real Banach space of (real) 2π-periodic functions $f(x) \in L^2(0, 2\pi)$ modulo constants with norm

$$(5) \qquad \|f\|^2_{\mathrm{BMO}} = \sup\{U_{f^2}(z) - [U_f(z)]^2 \colon z \in \mathbf{D}\} < \infty,$$

where $U_f(z)$ is the Poisson integral of $f(x)$. It is easily seen that

$$(6) \qquad U_{f^2}(z) - [U_f(z)]^2 = \frac{1}{2\pi} \int_0^{2\pi} [f(t) - U_f(z)]^2 \, P(t, z) \, dt,$$

where $P(t, z)$ is the Poisson kernel

$$P(t, z) = \frac{1 - |z|^2}{|e^{it} - z|^2} = \frac{\partial}{\partial n_\varsigma} \log \left| \frac{\varsigma - z}{1 - \bar{z}\varsigma} \right|$$

(n_ς is the outward normal to $\partial \mathbf{D}$ at $\varsigma = e^{it}$).

REMARK. The norm (5) is sometimes called the *Garsia norm*; it is equivalent to the original Nirenberg BMO norm (see, e.g., [**4**, p. 329]).

THEOREM 1. *Let $u \in B$ and $f_r(x) = u(re^{ix})$. Then*

$$(7) \qquad \|f_r\|_{\mathrm{BMO}} \leq 2^{-1/2} \|u\|_B \sqrt{|\log(1 - r^2)|} \qquad (0 < r < 1).$$

PROOF. By Green's formula ($\varsigma = e^{it}$),

$$
\begin{aligned}
U_{f_r^2}(z) - [U_{f_r}(z)]^2 &= \frac{1}{2\pi} \int_0^{2\pi} [u(re^{it}) - u(rz)]^2 \frac{\partial}{\partial n_\varsigma} \log \left| \frac{\varsigma - z}{1 - \bar{z}\varsigma} \right| \, dt \\
&= \frac{r^2}{\pi} \iint_D - \log \left| \frac{w - z}{1 - \bar{z}w} \right| |\operatorname{grad} u(rw)|^2 \, dA_w \\
&\leq \frac{r^2 \|u\|^2_B}{\pi} \iint_D - \log \left| \frac{w - z}{1 - \bar{z}w} \right| (1 - r^2 |w|^2)^{-2} \, dA_w,
\end{aligned}
$$

where dA_w is the Euclidean area element. The latter integral represents a superharmonic function $V(z)$ in $\mathbf{D}$ such that $V(z) = V(|z|)$ and $V(1^-) = 0$. Therefore, $\max\{V(z) \colon z \in \mathbf{D}\} = V(0)$ and

$$
\begin{aligned}
\|f_r\|^2_{\mathrm{BMO}} &\leq \frac{r^2 \|u\|^2_B}{\pi} \iint_D \log \frac{1}{|w|} (1 - r^2 |w|^2)^{-2} \, dA_w \\
&= 2r^2 \|u\|^2_B \int_0^1 |\log \rho| (1 - r^2 \rho^2)^{-2} \rho \, d\rho \\
&= \frac{1}{2} \|u\|^2_B |\log(1 - r^2)|,
\end{aligned}
$$

which completes the proof.

REMARK. The example $u(re^{ix}) = r \operatorname{Cos} x$ shows that the constant $2^{-1/2}$ in (7) is sharp. The example

$$u(re^{ix}) = \sum_{n=1}^{\infty} r^{k^n} \operatorname{Cos}(k^n x),$$

where $k \geq 2$, shows that $\sqrt{|\log(1 - r^2)|}$ cannot be replaced by a function of slower growth (cf. [**1**]).

3. Exponential estimate.

THEOREM 2. *There are positive numerical constants c and C such that for all $u(z) \in B$, $u(0) = 0$,*

$$(8) \qquad \int_0^{2\pi} \exp\left\{ \frac{c\,|u(re^{ix})|}{\|u\|_B \sqrt{|\log(1 - r^2)|}} \right\} dx \leq C \qquad (0 < r < 1).$$

PROOF. By the John-Nirenberg theorem [3] the distribution function $m(y)$ of an $f \in$ BMO satisfies $(y \geq 0)$

$$(9) \qquad \begin{aligned} m(y) &= \mathrm{meas}\{x \in (0, 2\pi)\colon |f(x) - f_{(0,2\pi)}| > y\} \\ &\leq 2\pi \exp\{-\gamma y / \|f\|_{\mathrm{BMO}}\}, \end{aligned}$$

where $f_{(0,2\pi)}$ is the average value of f on $(0, 2\pi)$ and γ is a positive numerical constant. Assuming $f_{(0,2\pi)} = 0$, (9) implies

$$(10) \qquad \int_0^{2\pi} \exp\left\{ \frac{\gamma\,|f(x)|}{2\,\|f\|_{\mathrm{BMO}}} \right\} dx \leq C,$$

where C is another numerical constant. Putting $f = f_r$ and applying (7), we obtain (8).

4. Radial growth.

THEOREM 3. *There is a numerical constant κ such that, for every $u(z) \in B$, $u(0) = 0$,*

$$(11) \qquad \limsup_{r \to 1^-} \frac{|u(re^{ix})|}{\sqrt{|\log(1 - r)|}\, \log|\log(1 - r)|} \leq \kappa \|u\|_B$$

for almost all x.

PROOF. (8) implies that $(0 < r < 1)$

$$\int_0^1 \frac{dr}{(1 - r)\log^2(e/1 - r)} \int_0^{2\pi} \exp\left\{ \frac{c\,|u(re^{ix})|}{\|u\|_B \sqrt{|\log(1 - r^2)|}} \right\} dx \leq C.$$

Therefore, for almost all x,

$$\int_0^1 \frac{1}{(1 - r)\log^2(e/1 - r)} \exp\left\{ \frac{c\,|u(re^{ix})|}{\|u\|_B \sqrt{|\log(1 - r^2)|}} \right\} dr < \infty,$$

which implies that

$$\lim_{r \to 1^-} \int_r^{(1+r)/2} (1 - \rho)^{-1} \log^{-2}(e/1 - \rho) \exp\left\{ \frac{c\,|u(re^{ix})|}{\|u\|_B \sqrt{|\log(1 - \rho^2)|}} \right\} d\rho = 0.$$

Putting $\mu(r, x) = \min\{|u(\rho e^{ix})|\colon r \leq \rho \leq (1 + r)/2\}$, we get

$$(12) \qquad \lim_{r \to 1^-} \left(\frac{c\mu(r, x)}{\|u\|_B \sqrt{|\log(1 - r)|}} - 2\log|\log(1 - r)| \right) = -\infty.$$

Thus, for almost all x, as $r \to 1^-$, $\mu(r, x)$ ultimately becomes

$$< c_1 \|u\|_B \sqrt{|\log(1 - r)|}\, \log|\log(1 - r)|,$$

with some other constant c_1. On the other hand, (1) shows that the oscillation of $u(\rho, x)$ on the interval $r \leq \rho \leq (1 + r)/2$ does not exceed $\|u\|_B$, which proves (11) with $\kappa = c_1$.

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