those who may be quite familiar with the general theory, will find much of interest. In sum, this is an interesting book, which well deserves the attention of those with an interest in the analytic side of several complex variables.

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Angeordnete Strukturen: *Gruppen*, *Körper*, *Projektive Ebenen*, by Sibylla Prieß-Crampe, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 98, Springer-Verlag, Berlin, 1983, ix + 286 pp., $71.00. ISBN 3-5401-1646-X

The study of (totally) ordered fields and groups is a fairly old discipline with venerable roots, the earliest contributions to which go back to the beginning of the century, with results by Hilbert, Hölder, and Hahn. Specifically, Hilbert (1899) considered a special ordered field of real-valued functions in order to establish the independence of certain axioms of geometry, Hölder (1901) showed that every archimedean ordered group can be embedded, as an ordered group, into the additive group of real numbers with its natural order, and Hahn (1907) proved that every ordered abelian group is isomorphic to an ordered subgroup of a certain group $H(\Gamma, \mathbf{R})$ of lexicographically ordered real-valued functions on some ordered set. The precise meaning of the latter result is as follows: Given an ordered set Γ, one considers those functions u: $\Gamma \to \mathbf{R}$ for which the support $S(u) = \{\gamma | u(\gamma) \neq 0\}$ is dually wellordered with respect to the ordering of Γ. It is immediate that these functions form a subgroup of the usual group of all real-valued functions on Γ, and that putting $u > 0$ iff $u(\alpha) > 0$ for $\alpha = \max S(u)$ then defines an ordered group, the so-called *Hahn group* $H(\Gamma, \mathbf{R})$. Hahn further showed that, whenever Γ is an ordered abelian group rather than merely an ordered set, $H(\Gamma, \mathbf{R})$ admits a multiplication, given by convolution: $u * v(\gamma) = \sum u(\alpha)v(\beta)$ $(\alpha\beta = \gamma)$, so that $H(\Gamma, \mathbf{R})$ becomes an ordered field. Note that for $\Gamma = \mathbf{Z}$, with the dual of its natural ordering, $H(\Gamma, \mathbf{R})$ is exactly the field of formal Laurent series $\sum c_n x^n$ $(n \gg -\infty)$ over $\mathbf{R}$, where an element is positive iff, in the usual order of the exponents $n \in \mathbf{Z}$, its first nonzero coefficient is positive.

The beauty and scope of its results make Hahn's 1907 paper a milestone in the subject, even if it was rather difficult to read. Moreover, the basic approach allowed for refinements and ramifications which were pursued in later years by a considerable number of authors. Typical examples would be the generalization of the Hahn Embedding Theorem to lattice-ordered groups by Conrad, Harvey, and Holland (1963) and the construction of more general fields of formal power series than the fields $H(\Gamma, \mathbf{R})$ for an abelian ordered group Γ in which $\mathbf{R}$ is replaced by an arbitrary ordered field K, convolution is modified by the intervention of a suitable factor system $(c_{\alpha\beta})_{\alpha,\beta \in \Gamma}$ on Γ with values in K so that $uv(\alpha) = \sum c_{\beta\gamma} u(\beta) v(\gamma)$ $(\beta\gamma = \alpha)$, and Γ is taken as an arbitrary ordered group. A naturally arising problem in this context is to see how appropriate properties of Γ and K determine desired properties of these ordered fields, and there are various interesting results concerning this.

A very substantial new idea was introduced into the subject by Artin and Schreier (1927) in their theory of formally real fields. The familiar property of $\mathbf{R}$ that $a_1^2 + a_2^2 + \cdots + a_n^2 = 0$ implies $a_1 = a_2 = \cdots = a_n = 0$ for any $a_1, a_2, \ldots, a_n \in \mathbf{R}$ is obvioulsy shared by any ordered field, and may therefore be viewed as a purely algebraic first order residue of the a priori not first order property of a field to possess a compatible ordering. A field which satisfies the stated identical implications is called *formally real*. The significance of this notion for the present topic is the remarkable fact that the formally real fields are indeed *exactly* the orderable fields, but this by no means exhausts its usefulness. It is natural, at least with the wisdom of hindsight, to pay special attention to those formally real fields which are "algebraically maximal" in the class of all such fields, i.e. which have no proper algebraic extension still formally real. These *real closed* fields, of which $\mathbf{R}$ is one, are obviously rather more similar to $\mathbf{R}$ than mere formally real fields, and a rich and attractive theory, beginning with the original work of Artin-Schreier, but still developing in the recent past, testifies to this fact. Among the classical results of particular importance are that every formally real field has an essentially unique real closure, that a real-closed field has exactly one compatible order, and that several facts of ordinary calculus hold for polynomials over real-closed fields. It is thus quite remarkable—although that is not the entire purpose of the enterprise—how much of what one considers analysis can in fact be captured in algebraic terms.

A further new concept, which has its own independent purpose but also plays an interesting role in the theory of ordered fields, is that of a *valuation* of a field K, by which is meant a homomorphism v of the multiplicative monoid of K into an ordered abelian group Γ, augmented by a zero 0 as smallest element, such that $v(a) = 0$ iff $a = 0$ and $v(a + b) \leqslant \max\{v(a), v(b)\}$ for all $a, b \in K$. For any valuation v of a field K, $A = \{a | v(a) \leqslant \varepsilon\}$ (ε the unit of Γ) is a subring of K such that $a \in A$ or $a^{-1} \in A$ for any $a \in K$. Such subrings of K are called *valuation rings*, and any of these determines a valuation of K in such a way that, up to a natural notion of isomorphism, valuations and valuation rings correspond to each other uniquely. The link with ordered fields is provided by the fact that any such field K has a natural valuation v which is *order compatible* in the sense that $0 < a \leqslant b$ implies $v(a) \leqslant v(b)$ for all a, $b \in K$. In general, the order compatible valuations of an ordered field are suggestively characterized as those for which the associated valuation ring is convex in K.

For any valuation v on a field K, the associated valuation ring A is local, i.e. has a unique maximal ideal M, and hence determines the field $K_v = A/M$, the associated *residue class field*. On the other hand, v gives rise to the *value group* Γ_v, the subgroup of Γ consisting of the $v(a)$, $a \neq 0$ in K. K_v and Γ_v are important features of the valuation v. Of particular interest are extension fields L of K with a valuation w extending a given valuation v on K such that the residue class field and the value group for v and w are the same. Such extensions are called *immediate*, and if there are no proper immediate extensions of K relative to v, one calls v a *maximal* valuation. There is a fairly elaborate theory involving these notions, such as characterizing maximal valuations in terms of the somewhat messy concept of *pseudoconvergence* due to Ostrowski 1935 (Kaplansky 1942), establishing the existence, and uniqueness if the residue class field has characteristic zero, of maximal immediate extensions (Kaplansky 1942), and characterizing properties of K by those of K_v and Γ_v for a given valuation v, especially a maximal one. Of special importance for the theory of ordered fields are valuations which are maximal and order compatible. Perhaps the most exciting result concerning these is Kaplansky's representation theorem which characterizes these, up to isomorphism (of the entire structure), as the formal power series fields given by Γ_v and K_v, with an appropriate factor system, with their natural order and valuation.

Yet another area with useful applications to ordered fields, as well as ordered groups, is topological algebra. Here, the connection arises from the fact that, in an ordered field K, say, the open intervals $]-\varepsilon, \varepsilon[$, $0 < \varepsilon$ in K, form a neighbourhood basis of 0 for a ring topology (and analogously for ordered abelian groups). The completion $\hat{K}$ of K with respect to this turns out to be a field again, as one might naively have hoped for, remembering that $\hat{\mathbf{Q}} = \mathbf{R}$. Again, with the latter in mind, it may not be surprising (although perhaps it should) that $\hat{K}$ as an ordered set can be identified with a certain part of the Dedekind completion δK of K as an ordered set: $c \in \delta K$ belongs to $\hat{K}$ iff for any $0 < \varepsilon$ in K there exist $a < c$ and $b > c$ in K such that $b - a \leqslant \varepsilon$. Needless to say, this permits an alternative construction of $\hat{K}$ in terms of special Dedekind cuts of K (Baer 1970, Massaza 1969/70). Natural questions now offer themselves in abundance: What properties of K determine desired properties of $\hat{K}$? How do completion $K \rightsquigarrow \hat{K}$ and real closure $K \rightsquigarrow \overline{K}$ interact? How are $\hat{K}$ and $\hat{L}$ related when K is an ordered subfield of L? What can be said about the order compatible valuations of K versus those of $\hat{K}$? And indeed, there is an extensive theory, the work of many authors, which provides interesting answers to questions of this type. In a similar vein one has questions of this sort for ordered abelian groups. Needless to say, the corresponding theory is not as rich as that for ordered fields, but it still has some interesting results.

A vast generalization of the notion of a field is that of *a ternary ring*, which is an algebraic system with a single ternary and two nullary operations subject to five axioms. Ternary rings arise as the coordinate rings of *projective planes*, where the latter can be described either as a 2-sorted structure (points, lines) with an appropriate relation between the two sorts (incidence), or as a set (of points) together with a collection of distinguished subsets (the lines) subject to the condition that any two distinct points belong to exactly one line, any two distinct lines have exactly one point in common, and there exist at least four points no three of which are collinear. Any field determines a ternary ring by defining the ternary operation as $T(a, x, b) = ax + b$, with 0 and 1 as the nullary ones, and the original field operations can, of course, be recovered from T. In this sense, fields are special ternary rings.

A projective plane is called *ordered* (Coxeter 1949) if each line has at least four points, and on each line a quaternary *separation* relation is given such that these relations are invariant under all perspectivities. Here, the defining properties of a separation relation express the way in which two of four different points on a circle separate the other two, and perspectivities are special maps derived from the basic structure of the plane. Any ordering of a projective plane $\mathfrak{E}$ induces a total order in the usual sense on any coordinatizing ternary ring K, making K into an *ordered ternary ring* in the sense that its operation T satisfies a number of laws of monotony (e.g. if $b < c$ then $T(a, x, b) < T(a, x, c)$). Conversely, if K is an ordered ternary ring then its order determines a separation relation on each line of $\mathfrak{E}$, making $\mathfrak{E}$ into an ordered projective plane, and the two correspondences are inverse to each other (Crampe 1958). This is a deep result and requires quite a complicated proof. The argument proceeds by considering, as one does for the coordinatization, the affine plane $\mathfrak{E}_l$, obtained by removing a line l from $\mathfrak{E}$, and the intervening notion of an ordered affine plane. Crucial in this context is the result (Crampe 1958) that the orderings of $\mathfrak{E}$ and $\mathfrak{E}_l$ uniquely correspond to each other.

With this basic setting one can ask which geometric properties of an ordered projective plane $\mathfrak{E}$ correspond to which properties of its ordered ternary coordinate rings K. Typical results: K is an ordered division ring (field) iff the Theorem of Desargue (Pappus) holds in $\mathfrak{E}$.

Besides ordered projective planes one has *topological projective planes*, meaning that each, the set of points and the set of lines, is equipped with a topology such that intersecting distinct lines and spanning a line by distinct points are continuous operations, and there is a nontrivial open set of points. The study of topological projective planes, some 30 years old, has produced a considerable number of interesting results but has also left some very natural open questions. Among the former are that the point space of a topological plane is regular Hausdorff, either connected or with trivial quasicomponents, second countable and σ-compact whenever it is locally compact, and compact whenever each line is compact. Further results deal with connectivity properties, especially with the question when the affine lines are homeomorphic to **R**. Regarding the relation between a projective plane $\mathfrak{E}$ and any of its ternary rings K, one has the rather obvious observation that if $\mathfrak{E}$ is topological then K is topological in the sense that its topology makes the fundamental ternary operation and certain further derived operations continuous. Interestingly, it is an open question, with only partial answers to date, whether every topological ternary ring actually arises this way.

Topological and ordered projective planes interact by the fact that any ordering of a projective plane $\mathfrak{E}$ determines topologies on points and lines making $\mathfrak{E}$ into a topological projective plane such that, on each line of $\mathfrak{E}$, the induced topology is the interval topology. A topological projective plane $\mathfrak{E}$ is then called *orderable* iff $\mathfrak{E}$ has an ordering such that the topology derived from the latter is the given topology of $\mathfrak{E}$. This notion is illuminated by the remarkable result (Einert 1975) that a connected projective plane is orderable iff all its affine lines are homeomorphic to **R**.

The concept of archimedean ordered field has its natural extension to ordered ternary rings, leading then to the notion of archimedean ordered projective plane. The fundamental result concerning these, highly sophisticated counterpart to the original result of Hölder's for archimedean ordered groups, is that they are exactly the ordered projective subplanes of projective planes whose affine lines are homeomorphic to **R** (Prieß-Crampe 1967). The proof follows the same idea as the classical construction of **R** from **Q** by means of Dedekind cuts, providing a (conditional) order completion for any archimedean ordered ternary ring, but is so vastly more involved that one can only wonder why.

All these topics, and a few others omitted in this survey, are treated in the book under review, the motivation for which the author attributes to a suggestion by Reinhold Baer some 10 years ago. No doubt this suggestion was a very fortuitous one, and the author has written a text of remarkable scope, wide erudition, and exemplary thoroughness, producing an admirable execution of the project conceived by the late master. As a comment, though by no means criticism, it might be observed that, at first sight, there appears to be some dichotomy between the parts that deal with groups and fields and those devoted to projective planes. The concerns regarding the former do lead, at times, in directions somewhat different from those regarding the latter. Still, bringing together this variety of different strands of ideas in the way presented here does make good sense, and the author has succeeded in providing a compelling picture of unity in diversity. Moreover, it is not unreasonable to expect that some thoughtful readers might find the picture, so carefully laid out, fertile ground for new ideas leading to interesting new work. What better can be said of a book of this kind?

P.S. There may be the odd misprint here or there, but there is one rather prominent one the reviewer feels duty bound to report: the title on the spine of the book reads "Angewandte Strukturen" = applied structures, which of course should be "Angeordnete Strukturen" = ordered structures. It is true that augmenting abstract algebra by order, or topology, is usually felt to make it less abstract, but "applied" may be going a bit too far....

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