Those who wonder why such a knowledgeable author has suddenly appeared with no previous record of publications will find that he is a relative of N. Bourbaki, if they inquire in the right circles.

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Percolation theory for mathematicians, by Harry Kesten, Progress in Probability and Statistics, vol. 2, Birkhauser, Boston, Mass., 1982, 423 pp., $30.00. ISBN 3-7643-3107-0

Physicists have enthusiastically embraced percolation models, and a dramatic explosion of physics literature on percolation has occurrred in recent years. This literature is rich in simulations, conjectures, heuristic methods, and a wide variety of applications and variations of the basic models. Mathematicians who experience frustration in tracing the thread of fact through this tangle of conjecture and empirical evidence will appreciate the mathematical rigor in *Percolation theory for mathematicians.*

Percolation models originated in discussions between Broadbent and Hammersley (1957) on the excluded volume problem in polymer chemistry and the design of coal miners' masks. Such topics suggested a probabilistic model for fluid flow in a medium with randomness associated with the medium rather than the fluid. Hence, percolation theory arose as an alternative to the more familiar diffusion models, in which randomness is associated with the fluid.

Models. In a percolation model, the medium is represented by a graph G, which is usually an infinite graph with some regularity of structure. Familiar examples are the square, triangular, and hexagonal lattices in two dimensions, and the cubic lattice in three dimensions. The fluid flow is determined by a random network of vertices and edges in the graph.

The random mechanism may be associated with either the vertices or the edges, so two standard models arise: In the *bond* percolation model, each edge is "occupied" by fluid with probability p and " vacant" with probability $1 - p$, independently of all other edges. In the *site* percolation model, each vertex is occupied with probability p and vacant with probability $1 - p$, while an edge is occupied only if both its endpoint vertices are occupied. Actually, the bond percolation model on a graph G may be transformed into an equivalent site percolation model on a graph G^c, known as the line graph or covering graph of G. For this reason, *Percolation theory for mathematicians* is set in the more general framework of site percolation models.

Critical probability. The extent of fluid flow possible from a single source site depends on the parameter p, and description of this dependence is the goal of the theory. While the flow is local when p is near zero, fluid spreads throughout the medium when p is near one.

The central concept in percolation theory is the critical probability, which is a threshold at which the transition from local flow to penetration of the medium occurs. Physicists studying phase transitions are greatly interested in this threshold phenomenon, which is uncharacteristic of diffusion models. In modeling a dilute ferromagnet, the critical probability represents the threshold between the presence or absence of spontaneous magnetization.

The use of different definitions of critical probability in the nonmathematical percolation literature has been a source of confusion. There are two definitions of primary interest: For a fixed site v in an infinite graph G, let C_v denote the number of sites wetted by fluid from a source at v. Denote the probability measure and expectation operator for the percolation model with parameter p by P_p and E_p respectively. The most common interpretation of the critical probability concept is in terms of existence of infinite clusters of occupied sites:

$$p_H = \inf\{\, p \in [0,1] : P_p(C_v) > 0\}.$$

The main alternative is based on the expected size of an occupied cluster:

$$p_T = \inf\{\, p \in [0,1] : E_p(C_v) = \infty\}.$$

If G is a connected graph, the definitions are independent of the choice of the source site v. While one intuitively expects that $p_H = p_T$, this is not always true, but $p_H \geqslant p_T$ easily follows from the definitions.

Duality. Harris (1960) derived a lower bound of $1/2$ for the square lattice bond model critical probability p_H, using the notion of a dual percolation model. For a planar graph G, the dual graph G^* is constructed by placing a site of G^* in each face of G and inserting a bond connecting each pair of sites of G^* which lie in neighboring faces of G. Each bond of G^* crosses exactly one bond of G, establishing a one-to-one correspondence between bonds of G and bonds of G^*. Given a bond percolation model on G, create the dual model on G^* by assigning each bond of G^* to be occupied or vacant if the bond it crosses is occupied or vacant respectively. If G is the square lattice, the dual model is also a square lattice bond percolation model with the same parameter value, a situation described as "self-duality". Since an edge is in a finite cluster of open edges if and only if it is surrounded by a circuit of closed edges in the dual model, the construction of circuits is important in determining critical probabilities in planar graph models.

Matching. Sykes and Essam (1964) derived critical probability values for a few planar lattice bond percolation models by a heuristic method. Motivated by the use of duality by Harris, they identified the concept of a matching graph, which plays the same role for site models. Suppose G is constructed from a planar graph M by inserting all the diagonal bonds in each face F in a subset $\mathscr{F}$ of faces of M. The matching graph of G, denoted G^*, is constructed from M by inserting all diagonal bonds in faces F which are not in $\mathscr{F}$.

For a rectangular region R, Sykes and Essam defined a function $\lambda_R(p)$ as the expected number of connected clusters of occupied sites in R divided by the number of sites in R. Taking the thermodynamic limit as R expands through a sequence of rectangular regions, the "clusters-per-site function" $\lambda(p)$ for the graph G is obtained. Reasoning using Euler's law concludes that $\lambda(p)$ and $\lambda^*(1-p)$ differ by a polynomial, where λ^* is the clusters-per-site function for the matching graph G^*. If singularities of λ and λ^* exist, they must occur at complementary values. Assuming that the clusters-per-site function for each graph has a unique singularity, Sykes and Essam defined a critical probability to be the location of the singularity. Their argument then implies that critical probabilities of matching graphs sum to one.

Since the covering graphs of a dual pair of planar graphs form a matching pair, the critical probabilities of bond models on a dual pair of graphs also sum to one. Thus, the critical probability of a site model on a self-matching graph or the bond model on a self-dual graph should equal $1/2$. The method did produce the correct common value for p_H and p_T of $1/2$ for the triangular lattice site model and the square lattice bond percolation model, and $2\sin(\pi/18)$ and $1 - 2\sin(\pi/18)$ for the triangular and hexagonal lattice bond percolation models, respectively. However, self-matching graphs exist for which the site percolation critical probabilities p_H and p_T are not $1/2$, and for which the clusters-per-site function has no singularity.

Revival. In contrast to the activity in physics, after the initial flurry of work in the late 1950s and early 1960s, mathematical percolation theory lay dormant until a revival in the late 1970s. Significant progress was made by Seymour and Welsh (1978). Considering the square lattice bond model, they let $S_n(p)$ denote the probability that a path of occupied bonds crosses the rectangle $[1, n+1] \times [1, n]$ from left to right. A critical probability p_S, based on crossing probabilities, was defined by $p_S = \inf\{p \in [0,1]\colon \limsup S_p(n) > 0\}$. Either an occupied path crosses a rectangle from left to right, or a vacant path crosses the dual rectangle from top to bottom, so, by self-duality,

$$S_n(p) + S_n(1-p) = 1.$$

Intricate arguments constructing circuits used this fact to bound the probability of existence of circuits, leading to the results that $p_H + p_T = 1$ and $p_T = p_S$. A key lemma in the circuit construction required that the model have two axes of symmetry, although it is not clear if any symmetry is necessary. While Seymour and Welsh did not determine the critical probability, the crossing probability concept provided the breakthrough that led to a rigorous determination. Using crossing probability techniques, Kesten (1980) supplied a clever method of sequentially closing bonds to cut occupied paths across the rectangles when $p < 1/2$, thereby evaluating p_S for the square lattice, concluding that

$$p_S = p_T = p_H = 1/2.$$

Sykes' and Essam's values for the triangular and hexagonal lattice bond models have been rigorously verified by Wierman (1981).

Contributions. Kesten's results are proved in the setting of multiparameter site percolation models on periodic graphs. In a multiparameter percolation model, the vertices of G are partitioned into sets which have different parameters representing their occupation probabilities. The "percolative region" in a multiparameter model is the subset of the parameter space corresponding to models where fluid penetrates the medium. The boundary of the percolative region is the "critical surface", which is the counterpart of the critical probability in a one-parameter model. The essential feature of a periodic graph is that it is invariant under translation by each member of a set of basis vectors for the space. In *Percolation theory for mathematicians*, Kesten characterizes the percolative region for multiparameter site models on two-dimensional periodic graphs with one axis of symmetry. The principal theorems are proved in Chapters 5–7. Chapter 6 relaxes the symmetry requirement from two axes to one, a major advance.

For a broad class of graphs, the principal theorems provide a rigorous verification of Sykes' and Essam's claim that critical probabilities of matching graphs sum to one (for the p_H, p_T, and p_S definitions). They allow the explicit determination of the critical surface in several multiparameter models. However, the results still do not provide explicit values of the critical probability for a wide class of graphs.

One of the few dimension-free results in percolation, that the critical probabilities p_T and p_S are equal for a periodic graph of any dimension, is proved in Chapter 5. This previously published result was proved using a block renormalization approach. Renormalization methods are popular in the physics literature, but have led to few other rigorous results in percolation.

Empirical evidence suggests that the probability of existence of an infinite occupied cluster or the expected size of an occupied cluster behave as powers of $p - p_H$. The exponents, if they exist, are known as "critical exponents". Such "critical exponents" are estimated in the physics literature by renormalization methods, but their existence is an open question. Chapter 8 presents upper and lower bounds for the exponent for square lattice models.

Other new results for percolation models are treated in the remaining chapters. In Chapter 9 the Sykes and Essam clusters-per-site function is shown to exist and proved to be analytic in p for $p \neq p_H$ for matching pairs of graphs to which the principal theorems apply. However, it has not yet been shown that a singularity exists at p_H for any periodic graph. A new result in Chapter 10 provides conditions under which the critical probability of a subgraph is strictly greater than the critical probability of the original graph. Asymptotic results for the resistance of a random electrical network are given in Chapter 11.

The book. *Percolation theory for mathematicians* is an excellent research monograph, presenting significant new results at the forefront of the theory. Even so, the material is accessible with a background of only one term of a graduate course in probability theory. Some acquaintance with graph theory would be helpful, but is certainly not essential. However, since the arguments are quite intricate, considerable mathematical sophistication is an important prerequisite. In fact, Kesten warns that the reader "will surely become impatient with the unpleasant details".

For a comfortable introduction to the field, a nonexpert will find the important tools and techniques, and statements and applications of the principal theorems, in Chapters 1–4. The reader may then test his mastery of this material by attempting selections from the other chapters. Such exploration may consist of the dimension-free result in §5.1, the rigorization of the Sykes and Essam singularity approach in Chapter 9, and the comparison of bond and site model critical probabilities in §10.1.

The more ambitious reader may also include the rest of Chapter 5, §6.1, and Chapter 7 for a basic understanding of the proof of the principal theorems. Chapter 6 is written primarily for researchers in percolation theory, so nonexperts are well advised to content themselves with the case having two symmetry axes (in §6.1), or perhaps the original treatment by Seymour and Welsh (1978).

Chapter 12 presents a collection of unsolved problems, which can be appreciated after browsing the opening paragraphs of the remaining chapters, which are intended for experts. The book provides complete subject and author indices, and a very useful list of repeated symbols and notation, citing the pages on which definitions are found.

Future directions. Although much has been accomplished in recent years, many stimulating research directions remain in percolation theory. Current knowledge is confined almost entirely to two-dimensional models, where critical probabilities are known for only a few graphs. The determination of critical probabilities of additional two-dimensional models and of higher-dimensional models, the existence and values of critical exponents, the existence and nature of singularities of the clusters-per-site function, and the application of renormalization methods all provide challenging research problems.

Additional research questions arise from the proliferation of models. The first-passage percolation model, where bonds are assigned random travel times and the asymptotic rate of spread of fluid is studied, was introduced by Hammersley and Welsh (1965) and is surveyed in the monograph by Smythe and Wierman (1978). The theory of subadditive stochastic processes originated in first-passage percolation theory. Directed percolation models, in which fluid may flow in only one fixed direction on each occupied bond, have connections with interacting particle systems models. Mixed models, with both bonds and sites random, are considered also, as well as models with randomly oriented bonds.

Conclusion. It is fitting that the first major book on percolation theory be authored by Harry Kesten, the major contributor to the field in recent years. The expert in the field will find this book indispensable, while it supplies a good introduction for the nonexpert. It is not intended to be a reference volume covering the entirety of percolation theory, but is limited to rigorous results for the classical models. In fact, so much has happened in recent years that it would be difficult to produce a single volume which is both rigorous and comprehensive. The monograph will help consolidate and unify the theory of the classical percolation models, although a tidy systematic understanding still lies in the distant future, considering the major unsolved problems and many loose ends that remain. Certainly the volume deserves the title of the series: "Progress in Probability and Statistics".

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Geometric aspects of convex sets with the Radon-Nikodým property, by Richard D. Bourgin, Lecture Notes in Math., vol. 993, Springer-Verlag, Berlin, 1983, xii + 474 pp., $22.00. ISBN 3-5401-2296-6

In the Spring of 1973, Jerry Uhl and I were putting the finishing touches on a manuscript entitled, *Vector measures*. With his usual wisdom, Jerry suggested we put the manuscript aside for six months or so. His reasoning went more or less like this: we're happy with what we've done *now*, so, if it still looks good to us in six months—all the better; besides, maybe something beautiful will happen in the meantime that really ought to be included. Jerry and I are unabashed optimists and so Jerry's suggestions offered extremely appealing alternatives.