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INTEGRAL REPRESENTAIONS ON HERMITIAN MANIFOLDS: THE $\bar\partial$-NEUMANN SOLUTION OF THE CAUCHY-RIEMANN EQUATIONS[1]

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1. Introduction. Let D be a relatively compact domain in a Hermitian manifold X of complex dimension n. The Cauchy-Riemann operator $\bar\partial$ extends to a densely defined operator

$$\bar\partial\colon L^2_{0,q}(D) \to L^2_{0,q+1}(D), \qquad 0 \le q \le n.$$

The inner product in $L^2_{0,q}(D)$ is given by

$$(f,g)_D = \int_D f \wedge *\bar g,$$

where $*$ is the Hodge operator defined by the Hermitian structure. If $\bar\partial^*$ is the Hilbert space adjoint of $\bar\partial$, one defines the complex Laplacian by

$$\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

Its significance for complex analysis lies in the fact that if $Nf \in \operatorname{dom}\Box$ solves $\Box(Nf) = f$ and $\bar\partial f = 0$, then $u = \bar\partial^* Nf$ is the unique solution of $\bar\partial u = f$ which is orthogonal to $\ker\bar\partial$. J. J. Kohn has established existence and regularity properties of the solution operator N, giving the solution of minimal norm—the so called $\bar\partial$-Neumann operator—in case D is strictly pseudoconvex [**5**], and in more general cases as well [**6**]. The proofs are based on a priori estimates in L^2-Sobolev spaces, and therefore they do not give any explicit information about the kernels of N or $\bar\partial^*N$.

In recent years there has been much interest in finding more explicit and concrete representations of the abstractly defined operators N and $\bar\partial^*N$ (see [**2, 3, 9, 10, 12**]). In [**7**] we began to study $\bar\partial^*N$ by using the calculus of Cauchy-Fantappié kernels in $\mathbf{C}^n$, in analogy to the work of Kerzman and Stein [**4**] and Ligocka [**8**] for the Szegö, respectively, Bergman kernel; in contrast to the scalar case, the incompatibility of the Euclidean metric with the complex geometry of the boundary of D turned out to be a major obstruction in the general case.

In the present paper we overcome this obstruction by generalizing the results in [**7**] to arbitrary Hermitian manifolds; this enables us to then introduce a special Levi metric—similar to the one in [**2**]—and to establish the required symmetry properties of the kernels. Our main result gives a new and completely explicit integral representation of the principal part of $\bar\partial^*N$ on

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range $\overline{\partial}$. It is likely that these methods will lead to a corresponding representation of N itself (see [**11**] for the case of the unit ball in $\mathbf{C}^n$).

2. Kernels on Hermitian manifolds. Given a Hermitian manifold (X, ds^2) of dimension $n > 1$, we fix a function ρ on $X \times X$ which agrees with the geodesic distance function in a neighborhood U of the diagonal Λ, and which is positive and C^∞ on $X \times X - \Lambda$. Generic error terms will be denoted by $\mathcal{E}_j$, $j \in Z$, meaning that $\mathcal{E}_j$ is a double form which is smooth off the diagonal on a region in $X \times X$, and which satisfies $|\mathcal{E}_j| \lesssim \rho^j$ locally near the diagonal; integral operators with kernels $\mathcal{E}_j$ will also be denoted by $\mathcal{E}_j$.

Define the double form Γ_q on $X \times X - \Lambda$ by

$$\Gamma_q = \frac{(n-2)!(-1)^q}{q! 2^{1+q} \pi^n} \frac{(\overline{\partial}_x \partial_y \rho^2)^q}{\rho^{2n-2}}, \qquad 0 \leq q \leq n.$$

Since $\Box = \frac{1}{2}\Delta +$ terms of order ≤ 1, standard results in Riemannian geometry (see de Rham [**1**]) and integration by parts imply the following representation formula (see [**7**] for the case $X = \mathbf{C}^n$).

LEMMA 1. *If $D \Subset X$ has C^1 boundary and $f \in C^1_{0,q}(\overline{D})$ satisfies the first $\overline{\partial}$-Neumann boundary condition, then*

$$f(y) = -\int_{\mathrm{b}D} f \wedge *\partial_x \overline{\Gamma}_q + (\overline{\partial} f, \overline{\partial}_x \Gamma_q)_D + (\overline{\partial}^* f, \vartheta_x \Gamma_q)_D + (f, \mathcal{E}_{1-2n})_D$$

for $y \in D$.

For a double form W on a subset of $X \times X$, of type $(1,0)$ in x and type $(0,0)$ in y, one defines the generalized Cauchy-Fantappié kernels $\Omega_q(W)$ by

$$\Omega_q(W) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} W \wedge (\overline{\partial}_x W)^{n-q-1} \wedge (\overline{\partial}_y W)^q.$$

It follows that

$$- * \partial_x \overline{\Gamma}_q = \Omega_q(\partial_x \rho^2/\rho^2) + \mathcal{E}_{2-2n}.$$

Now suppose that $D \Subset X$ is strictly pseudoconvex with C^∞ boundary. Fix a defining function $r \in C^\infty(X)$ for D which is strictly plurisubharmonic in a neighborhood of bD. By suitably patching functions defined in local coordinates as in the case $D \subset \mathbf{C}^n$ (see [**4** or **7**]), one constructs a smooth function ϕ on $\overline{D} \times \overline{D}$ with $\phi(x,x) = 0$ for $x \in \mathrm{b}D$, $\phi(x,y) \neq 0$ if $x \notin \mathrm{b}D$ or $y \neq x$, $\phi(x,y)$ holomorphic in y for x near bD and $\rho(x,y) < \varepsilon$, and $\phi(x,y) = -\overline{\phi(y,x)} = \mathcal{E}_3$. If $W = \partial_x \phi/\phi$ and $B = \partial_x \rho^2[\rho^2 + 2r(x)r(y)]^{-1}$, let

$$\begin{aligned} A_q &= A_q(W,B) \\ &= \frac{1}{(2\pi i)^n} \sum_{\mu=0}^{n-q-2} a_{\mu,q} W \wedge B \wedge (\overline{\partial}_x W)^\mu \wedge (\overline{\partial}_x B)^{n-q-\mu-2} \wedge (\overline{\partial}_y B)^q \end{aligned}$$

for $0 \leq q \leq n-2$, and 0 otherwise, with suitably chosen rational constants $a_{\mu,q}$. Set $L_q = (-1)^{q+1} * \overline{A}_q$ and define $T_q \colon L^2_{0,q+1}(D) \to L^2_{0,q}(D), 0 \leq q < n$, by

$$T_0 g = (g, \vartheta_x L_0 - *\overline{\Omega_0(W)} + \overline{\partial}_x \Gamma_0)_D,$$
$$T_q g = (g, \vartheta_x L_q - \partial_y L_{q-1} + \overline{\partial}_x \Gamma_q)_D \quad \text{for } q \geq 1.$$

An analysis of the kernels as in the case $X = \mathbf{C}^n$ shows that T_q is "smoothing of order 1/2", that is, T_q is bounded from L^∞ into $\Lambda_{1/2}$.

3. Main results. The Hermitian metric ds^2 on X is said to be a *Levi metric* for D (or rather r) if ds^2 is conformally equivalent to

$$\sum \left(\frac{\partial^2 r}{\partial z_j \, \partial \bar z_k} \right) dz_j \otimes d\bar z_k$$

in a neighborhood of bD.

THEOREM 1. *Let ds^2 be a Levi metric for D, normalized so that $\|\partial r\|_{ds^2} = 1$ near* bD. *Then T_q is the principal part of $\bar\partial^* N$ on the range of $\bar\partial$: $L^2_{0,q}(D) \to L^2_{0,q+1}(D)$.*

REMARK. If D is the unit ball in $\mathbf{C}^n$ with the Euclidean metric, $\sqrt{2}r = 1 - |x|^2$, and $\phi = 1 - \langle y, x\rangle$, then $T_q \equiv \bar\partial^* N$ on the whole space $L^2_{0,q+1}$ (see [**11**]). It appears likely that the restriction to range $\bar\partial$ in Theorem 1 is unnecessary.

Theorem 1 is a consequence of the following fundamental integral representation formula on strictly pseudoconvex domains, valid for arbitrary metrics, and Theorem 3. We denote by $\mathfrak{A}_j$ a generic integral operator whose kernel is admissible of weighted order $\geq j$ (as defined in [**7**]).

THEOREM 2. *A form $f \in C^1_{0,q}(\bar D) \cap \operatorname{dom} \bar\partial^*$ has the representation*

$$\begin{aligned} f = {} & T_q \bar\partial f + T^*_{q-1} \bar\partial^* f + E_q f \\ & + \mathcal{E}_{1-2n} f + \mathcal{E}_{2-2n}(\bar\partial f, \bar\partial^* f) + \mathfrak{A}_1 f + \mathfrak{A}_2(\bar\partial f, \bar\partial^* f), \end{aligned}$$

where

$$E_0 f = (f, *\partial_x \overline{\Omega_0(W)})_D$$

and

$$E_q f = (f, \vartheta_x \partial_y L_{q-1} - (\vartheta_x \partial_y L_{q-1})^*)_D \quad \text{for } q \geq 1.$$

THEOREM 3. *If ds^2 is a Levi metric, normalized as in Theorem* 1, *then E_q is admissible of weighted order ≥ 1 for all $q \geq 1$.*

The proof of Theorem 2 involves Lemma 1 and a generalization of the calculus of Cauchy-Fantappié forms in $\mathbf{C}^n$ to Hermitian manifolds. Theorem 3 is based on a delicate analysis of the leading terms of $\vartheta_x \partial_y L_{q-1}$; since these are of weighted order ≥ 0, but not ≥ 1 in general, the main point is a cancellation of singularities due to certain symmetries of the kernels. The result holds for arbitrary metrics in case $q = n - 1$, but if $1 \leq q < n - 1$, the Levi metric condition is essential.

We conclude by stating one of the many applications of these results.

THEOREM 4. *Let ds^2 be a Levi metric for D. For $q \geq 1$ and $f \in L^2_{0,q} \cap \operatorname{dom} \bar\partial \cap \operatorname{dom} \bar\partial^*$, one has*

(i)

$$\|f\|_{\Lambda_{1/2}} \lesssim \|f\|_{L^2} + \|\bar\partial f\|_{L^\infty} + \|\bar\partial^* f\|_{L^\infty};$$

and

(ii)

$$\|\bar{\partial}^* N f\|_{\Lambda_{1/2}} \lesssim \|f\|_{L^\infty}, \quad \text{if } f \text{ is } \bar{\partial}\text{-exact}.$$

Theorem 4(i) is the analogue in Hölder norms of Kohn's basic estimate. The corresponding version for $q = 0$ is

(i_0)

$$\|f - P_0 f\|_{\Lambda_{1/2}} \lesssim \|\bar{\partial} f\|_{L^\infty},$$

where $P_0\colon L^2_{0,0} \to L^2_{0,0} \cap \mathcal{O}$ is the orthogonal projection. Estimate (i_0) holds for arbitrary metrics; it follows from Theorem 2 and symmetry properties of E_0 (cf. [**7**]); it was first proved in [**2**] by different methods for Levi metrics, and in [**8**] by the above methods for $X = \mathbf{C}^n$ with the Euclidean metric. Different proofs of Theorem 4 have been announced in [**9** and **10**], but, to our knowledge, detailed proofs have not been published.

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