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SECONDARY CLASSES AND TRANSVERSE MEASURE THEORY OF A FOLIATION

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1. The purpose of this note is to announce several theorems showing how the secondary classes of a foliation $\mathcal{F}$ of a compact manifold X depend upon the measure theoretic properties of the equivalence relation determined by the foliation. The relevant properties are:

(i) amenability [**14**], which is equivalent to hyperfiniteness by Connes-Feldman-Weiss [**3**]; and

(ii) the Murray-von Neumann type.

A set $B \subset X$ is *saturated* if it is the union of leaves of $\mathcal{F}$. The equivalence relation $\mathcal{F}$ has *type* I if there is a measurable subset of X which intersects almost every leaf exactly once; *type* II if it admits an invariant measure, finite or infinite in the given measure class but does not have an essential saturated set of type I; and *type* III if it does not have any essential saturated sets of types I or II. Every equivalence relation can be decomposed into parts of types I, II, and III. These types correspond to certain algebraic properties of the von Neumann algebra $\mathcal{M}(X, \mathcal{F})$ associated with the equivalence relation [**1, 13**].

Let X be a compact manifold without boundary and $\mathcal{F}$ a C^2, codimension-n foliation of X. The secondary classes are given by a map $\Delta_*\colon H^*(\mathrm{WO}_n) \to H^*(X; \mathbf{R})$ with image spanned by the classes of the form $\Delta_*(y_I c_J)$. Here, y_I is a basis element for the relative cohomology $H^*(\mathrm{gl}_n, \mathrm{O}_n)$, and c_J is a Chern form of degree at most $2n$. If degree $c_J = 2n$, we say the class is *residual.* The Godbillon-Vey classes are those of the form $\Delta_*(y_1 c_J) \in H^{2n+1}(X; \mathbf{R})$, with $y_1 \in H^1(\mathrm{gl}_n, \mathrm{O}_n)$, the normalized basis element. The generalized Godbillon-Vey classes are those of the form $\Delta_*(y_1 y_I c_J)$, where $y_I = 1$ is permitted. (For a convenient reference, see [**11**].)

The residual secondary classes have the unusual property that they localize to the measurable saturated subsets of X: for each such $B \subset X$ and residual $y_I c_J \in H^p(\mathrm{WO}_n)$, the restriction $\Delta_*(y_I c_J)|B \in H^p(X)$ is well defined [**5**]. The following theorems are stated for the secondary classes of $\mathcal{F}$ on X, but corresponding theorems also hold for the localized classes $\Delta_*(y_I c_J)|B$ of the restriction $\mathcal{F}|B$.

THEOREM 1. *If $\mathcal{F}$ has type* I, *then all residual secondary classes of $\mathcal{F}$ are zero.*

Since $\mathcal{F}$ has type I if and only if it is a fibration in the category of measurable equivalence relations, Theorem 1 generalizes the well-known fact that the secondary classes are zero for a smooth fibration.

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THEOREM 2. *Assume there is a generalized G-V class which is nonzero for $\mathcal{F}$. Then $\mathcal{F}$ contains a nontrivial component of type* III. *If $\mathcal{F}$ is ergodic, $\mathcal{M}(X, \mathcal{F})$ must be a type* III *factor.*

The factors of type III are divided into subtypes III_0, III_λ for $0 < \lambda < 1$ and III_1. An ergodic foliation of type III_λ has an absolutely continuous transverse measure which is multiplied by powers of λ under the holonomy maps. If this measure is given by a C^1 density, it is easy to see that the G-V class $\Delta_*(y_1 c_1^n)$ is zero [**6**]. One can speculate whether this holds for all type III_λ-foliations and what role the type III classification plays in the values of the G-V classes. For a recent development in the case of type III_0 foliations, see Connes [**2**].

THEOREM 3. *If $\mathcal{F}$ is amenable then all residual secondary classes $\Delta_*(y_I c_J)$ $\in H^{p+2n}(X; \mathbf{R})$ for $p > 1$ are zero.*

The original Rousarrie examples of foliations with $\Delta_*(y_1 c_1^n)$ nonzero are amenable, so Theorem 3 cannot be extended to the secondary classes of degree $2n + 1$ without additional hypotheses. This is discussed further in [**6**].

If almost every leaf of $\mathcal{F}$ has subexponential growth, then $\mathcal{F}$ is amenable. Combining Theorem 3 with Theorem 1 of [**6**] yields:

COROLLARY 4. *If almost every leaf of $\mathcal{F}$ has subexponential growth, then all residual secondary classes of $\mathcal{F}$ are zero.*

2. Given a generalized G-V class $y_I c_J \in H^*(\text{WO}_n)$, using Thurston's realization theorems and the topological methods of [**7, 9**] it is often possible to construct on a given compact orientable manifold X uncountably many distinct codimension-n foliations for which $\Delta_*(y_I c_J)$ takes distinct nonzero values. The above theorems then place restrictions on the possible dynamics of such $\mathcal{F}$ and the measure properties of its von Neumann algebra $\mathcal{M}(X, \mathcal{F})$. Our general expectation is that the residual secondary classes of $\mathcal{F}$ *directly* define invariants of $\mathcal{M}(X, \mathcal{F})$ and the C^*-algebra $C^*(X, \mathcal{F})$ (cf. [**2**]) and for $y_I c_J$ with $y_I \in H^*(\text{sl}_n, \text{O}_n)$ we conjecture these classes are invariants of orbit equivalence.

3. Sketch of proofs. For each cohomology class $y_I \in H^*(\text{gl}_n, \text{O}_n)$ there is a *Weil measure* $\chi(y_I)$ on the quotient measure space $X/\mathcal{F}$. These generalize the *Godbillon measure* $g = \chi(-2\pi \cdot y_1)$ introduced by Duminy to study the G-V class of codimension one foliations [**4**]. The residual secondary classes of $\mathcal{F}$ are determined by the values of the Weil measures, so it suffices to show the appropriate $\chi(y_I) = 0$. Theorems 1–3 are consequences of our study of how the measures $\chi(y_I)$ depend on the transverse dynamics of $\mathcal{F}$. Let $Q \to X$ denote the normal bundle to $\mathcal{F}$. The following is an essential tool for calculating the Weil measures.

THEOREM 5.[1] *Let h_0 be a measurable metric on $Q \to X$ whose restriction to each leaf of $\mathcal{F}$ is smooth. If h_0 has a uniform bound on all of X for the norms of the first derivatives of h_0 in leaf directions, then the Weil measure*

[1] This result was announced by the first author at the AMS Special Session on Foliations in East Lansing, Michigan, November 1982.

$\chi(y_I)$ can be calculated from the measurable family of closed forms $\{\chi(y_I)|L$ s.t. L is a leaf of $\mathcal{F}\}$ defined leafwise by means of the smooth metrics $\{h_0|L$ s.t. L is a leaf of $\mathcal{F}\}$.

To prove Theorem 1 we use the assumption that $\mathcal{F}$ has a measurable cross-section to show there is a metric h_0 on Q satisfying Theorem 5 and which is almost everywhere holonomy invariant. This implies the restricted classes $\chi(y_I)|L$ vanish a.e., so the measures $\chi(y_I) = 0$ and the theorem follows.

For Theorem 2 assume there is an absolutely continuous invariant transverse measure for $\mathcal{F}$ with almost every leaf essential. Then Theorem 5 implies the Godbillon measure $g = 0$ and all classes $\Delta_*(y_1 y_I c_J) = 0$, a contradiction.

For Theorem 3 we use a fundamental theorem of Zimmer [**14**] to conclude that the normal smooth $\mathrm{GL}_n(\mathbf{R})$-cocycle of an amenable foliation is measurably equivalent to a cocycle taking values in one of 2^n maximal amenable subgroups of $\mathrm{GL}_n(\mathbf{R})$. The key result is to show this cocycle can be made *tempered* or locally bounded in leaf directions so that Theorem 5 applies. This follows from some general results concerning cocycles over metric equivalence relations with values in linear Lie groups. More specifically, let $\mathcal{F}$ be a discrete equivalence relation provided with a measurable family of metrics on the leaves so that any ball contains finitely many elements and the number of those elements grow at most exponentially with the radius. We call such an object a metric equivalence relation of exponential type.

THEOREM 6. *Let $\varphi\colon \mathcal{F} \to H$ be a measurable cocycle from a metric equivalence relation of exponential type into a maximal amenable subgroup H of $\mathrm{GL}_n(\mathbf{R})$. If φ is measurably equivalent to a tempered cocycle with values in $\mathrm{GL}_n(\mathbf{R})$, it is also measurably equivalent to a tempered cocycle with values in H.*

This is one of the series of results concerning the asymptotic behavior and tempering of cocycles over group actions and metric equivalence relations. Other results will appear in [**10, 12**].

We then use a result of [**8**] to conclude that for y_I of degree $p > 1$, the class $\Delta_*(y_I)|L$ is exact for a.e. leaf L, *and* there is a global measurable *bounded* $(p-1)$-form on X implementing this. By the leafwise Stokes' Theorem [**5**] and Theorem 5, the Weil measure $\chi(y_I)$ is zero.

Full proofs of the above plus related results will appear in [**10**].

Our techniques suggest the following:

CONJECTURE 7. For $y_I \in H^*(\mathrm{sl}_n, \mathrm{O}_n)$, the Weil measure $\chi(y_I)$ is quasi-invariant under orbit equivalence: If $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ are orbit equivalent, then $\chi(y_I)$ assigns the value zero to corresponding sets in $X/\mathcal{F}$ and $X'/\mathcal{F}'$.

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