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TOEPLITZ OPERATORS AND SOLVABLE C^*-ALGEBRAS ON HERMITIAN SYMMETRIC SPACES

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Bounded symmetric domains (Cartan domains and exceptional domains) are higher-dimensional generalizations of the open unit disc. In this note we give a structure theory for the C^*-algebra $\mathcal{T}$ generated by all *Toeplitz operators* $T_f(h) := P(fh)$ with continuous symbol function $f \in \mathcal{C}(S)$ on the Shilov boundary S of a bounded symmetric domain D of arbitrary rank r. Here h belongs to the *Hardy space* $H^2(S)$, and $P\colon L^2(S) \to H^2(S)$ is the Szegö projection. For domains of rank 1 and tube domains of rank 2, the structure of $\mathcal{T}$ has been determined in [**1, 2**]. In these cases Toeplitz operators are closely related to pseudodifferential operators. For the open unit disc, $\mathcal{T}$ is the C^*-algebra generated by the unilateral shift.

The structure theory for the general case [**12**] is based on the fact that D can be realized as the open unit ball of a unique *Jordan triple system* Z [**7**, Theorem 4.1]. Denoting the Jordan triple product by $\{uv^*w\}$, a *tripotent* $e \in Z$ satisfies $\{ee^*e\} = e$. Tripotents generalize the partial isometries of matrix algebras and determine the *boundary structure* of $D \subset Z$ (cf. [**7**, Theorem 6.3]). Our principal result ([**12**]; cf. also [**3, 4, 8**]) is the following:

THEOREM 1. *The Toeplitz C^*-algebra $\mathcal{T}$ associated with a bounded symmetric domain $D \subset Z$ of rank r is solvable of length r, i.e. there exists a chain*

$$\{0\} = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_r \subset \mathcal{I}_{r+1} = \mathcal{T}$$

of closed two-sided ideals $\mathcal{I}_k$ such that for $0 \le k \le r$ there is a C^*-algebra isomorphism ("k-symbol")

$$\sigma_k\colon \mathcal{I}_{k+1}/\mathcal{I}_k \to \mathcal{C}(S_k) \otimes \mathcal{K}(H_k),$$

where S_k denotes the compact manifold of all tripotents $e \in Z$ of rank k and $\mathcal{K}(H_k)$ denotes the C^*-algebra of all compact operators on a Hilbert space H_k. Further, $\dim(H_k) = \infty$ *for* $k < r$ *and* $\dim(H_r) = 1$.

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COROLLARY. *The spectrum of $\mathcal{T}$ can be identified with the set of all tripotents of Z. The ideal $\mathcal{I}_r$ is the closed commutator ideal of $\mathcal{T}$ and $\mathcal{T}/\mathcal{I}_r \approx \mathcal{C}(S)$, where $S = S_r$ is the Shilov boundary. Further, $\mathcal{I}_1 = \mathcal{K}(H^2(S))$.*

The proof of Theorem 1 is based on a detailed study of the harmonic analysis in $H^2(S)$ [**10**] and of the fine structure of single Toeplitz operators [**11**]. Since the Toeplitz C^*-algebra $\mathcal{T}$ associated with a reducible bounded symmetric domain D can be realized as a tensor product, we may assume that D is irreducible. Let $\mathcal{P}(Z)$ denote the polynomial algebra on Z and let K be the largest connected group of biholomorphic automorphisms of D fixing the origin.

The next result [**10**], based on ideas from [**6**], applies to domains equivalent to a *tube domain* (generalized upper half-plane). In this case the Jordan triple system Z is actually a unital *Jordan algebra.*

THEOREM 2. *Suppose the domain D is of tube type. Then*

$$\mathcal{P}(Z) \approx \mathbf{C}[N] \otimes \mathcal{H}(Z),$$

where N denotes the norm function ("generalized determinant") of the Jordan algebra Z and $\mathcal{H}(Z)$ is the space of all harmonic polynomials (for the commutator subgroup of K).

In order to apply Theorem 2 to a general domain $D \subset Z$, consider for $1 \leq k \leq r$ the Jordan algebra $Z_k := \{z \in Z\colon \{ee^*z\} = z\}$ of rank k with unit element $e := e_{r+1-k} + \cdots + e_r$, where $\{e_1, \ldots, e_r\}$ denotes a *frame* of orthogonal minimal tripotents of the Jordan triple system Z [**7**, §5]. Denote by N_k the norm function of Z_k, viewed as a polynomial on Z. The Peter-Weyl decomposition of $H^2(S)$, determined in [**9**] and described case by case in [**5**], can now be realized as follows [**10**]:

THEOREM 3. *The irreducible K-module $E_m \subset \mathcal{P}(Z)$ with signature $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$ is generated by the conical polynomial $N_m = N_1^{l_1} N_2^{l_2} \cdots N_r^{l_r}$, where $m_k = l_k + \cdots + l_r$ for all k.*

The K-invariant scalar product $(u|v)$ on Z given by the generic trace [**7**, 4.15] induces a *differential scalar product* $(p|q)_Z$ for polynomials $p, q \in \mathcal{P}(Z)$ [**6**, III.1]. Let $(\ |\ )_S$ be the *integral scalar product* in $H^2(S)$. Using integral formulas for semisimple Lie groups, the relationship between these K-invariant scalar products can be computed explicitly [**11**]. Let (r, s, t) denote the *type* of D, defined via the Peirce decomposition of Z [**7**, Theorem 3.14].

THEOREM 4. *For every signature $m = (m_1, \ldots, m_r)$ and all $p, q \in E_m$, we have*

$$\frac{(p|q)_Z}{(p|q)_S} = \prod_{j=1}^{r} \frac{\left(m_j + \frac{1}{2}s(r-j) + t\right)!}{\left(\frac{1}{2}s(r-j) + t\right)!}.$$

As a consequence of Theorem 4, the fine structure of "polynomial" Toeplitz operators (generating $\mathcal{T}$) can be related to polynomial differential operators $h(z)(\partial/\partial z)$, where z denotes the "coordinate" of Z (cf. [**11**]):

THEOREM 5. *Suppose $l(z) = (z|v)$ is a linear form. Then*

$$T_l^*(p) = \sum_{j=1}^{r} \left(m_j + \frac{s}{2}(r-j) + t\right)^{-1} \left(\left(v\frac{\partial}{\partial z}\right)p\right)_{m-\varepsilon_j},$$

$$T_l(p) = \sum_{j=1}^{r} \left(m_j - \frac{s}{2}(j-1)\right)^{-1} \left(\left(\{zv^*z\}\frac{\partial}{\partial z}\right)p\right)_{m+\varepsilon_j}$$

for all $p \in E_m$, the subscript denoting the Peter-Weyl component for signature

$$m \pm \varepsilon_j = (m_1, \ldots, m_{j-1}, m_j \pm 1, m_{j+1}, \ldots, m_r).$$

COROLLARY. *The commutator $[T_l, T_l^*]$ is a "diagonal" operator respecting the Peter-Weyl decomposition of $H^2(S)$.*

Theorem 5 enables us to construct the *irreducible representations* of the Toeplitz C^*-algebra $\mathcal{T}$ [**11**]. For a tripotent $e \in Z$, the Jordan triple system $Z_e := \{w \in Z : \{ee^*w\} = 0\}$ contains the bounded symmetric domain $D \cap Z_e$ with Shilov boundary S_e. For $f \in \mathcal{C}(S)$ define $f_e \in \mathcal{C}(S_e)$ by $f_e(w) := f(e+w)$. Consider the "peaking functions"

$$h_e^i(z) := c_i(\exp(z|e))^i$$

for $i \geq 0$, where $c_i > 0$ is a constant such that $\|h_e^i\| = 1$.

THEOREM 6. *For each tripotent $e \in Z$ there exists an irreducible representation σ_e ("e-symbol") of $\mathcal{T}$ on the Hardy space $H^2(S_e)$, such that $\sigma_e(T_f) = T_{f_e}$ for all $f \in \mathcal{C}(S)$ and*

$$\lim_{i\to\infty} \|A(h_e^i \cdot q) - h_e^i \sigma_e(A)q\| = 0$$

for all $q \in \mathcal{P}(Z_e)$ and all operators A in a dense $*$-subalgebra of $\mathcal{T}$.

For $0 \leq k \leq r$ let $\mathcal{I}_k \subset \mathcal{T}$ be the joint kernel of all e-symbol homomorphisms σ_e for tripotents $e \in S_k$ of rank k. Theorem 1 now follows from the fact that $\mathcal{I}_1$ consists of *compact* operators. More generally, the ideals $\mathcal{I}_k$ have an internal characterization [**12**]:

THEOREM 7. *For $0 \leq k \leq r$ let P_k denote the orthogonal projection from $H^2(S)$ onto the Hilbert sum of all K-modules E_m satisfying $m_{k+1} = \cdots = m_r = 0$. Then $\mathcal{I}_{k+1}$ is the C^*-algebra generated by all operators $T_pP_kT_q^*$ for $p, q \in \mathcal{P}(Z)$.*

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