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Semimartingales, a course on stochastic processes, by Michel Métivier, de Gruyter Studies in Mathematics, Vol. 2, Walter de Gruyter & Co., Berlin, 1982, xi + 287 pp., DM 88.--, $40.00. ISBN 3-1100--8674-3

The semimartingale calculus has emerged from the general theory of processes as an important tool for what Métivier claims in his preface is "the goal of many: the description of stochastic systems, the ability to study their behavior and the possibility of writing formulas and computational algorithms to evaluate and identify them (without mentioning their optimization!)." We will first describe some important ingredients of the calculus, beginning with the martingale stochastic integral.

Let $(\Omega, \mathbf{F}, P)$ be a probability space and let $(\mathbf{F}_t\colon t \geqslant 0)$ be an increasing family of sub-σ-algebras of $\mathbf{F}$. A random process $M = (M_t)$ is a martingale if M_s is an $\mathbf{F}_s$ measurable random variable for each s (i.e., if M is adapted) and if $E[M_t|\mathbf{F}_s] = M_s$ whenever $t \geqslant s$. This definition makes sense even for Banach-space-valued random processes. On the other hand, it is rather tricky to define martingales with values in a manifold—but J. M. Bismut did it using localization and manifold connections [**22**]. The most famous martingale is the Wiener process, sometimes called brownian motion, $W = (W_t\colon t \geqslant 0)$. (See [**24**] for references omitted here.)

A stochastic integral

$$\int_0^t h(s, \omega)\, dM(s, \omega) \qquad \left(\text{abbreviated } \int_0^t h_s\, dM_s\right)$$

can be defined when M is a martingale for certain types of functions h. As the notation suggests, a stochastic integral is similar to a Stieltjes-Lebesgue integral for each ω in Ω. However, the notation is misleading—many interesting martingales, the Wiener process included, have sample paths of unbounded variation over any interval for ω in a set of probability one. This precludes defining the stochastic integral as a Stieltjes-Lebesgue integral for each ω.

A recipe for contructing stochastic integrals is the following. First, if h is piecewise constant in t with jumps at $t_1, t_2, \ldots, t_n$, it is natural to define the stochastic integral by

$$\int_0^t h_s\, dM_s = \sum_i h(t_i)\left(M_{t_{i+1}} - M_{t_i}\right).$$

If the value of the integral, which is a random variable for each t, is sufficiently continuous in h as h ranges over a suitable collection of step functions, then the integral can be defined for h in a larger collection of functions.

Stochastic integrals were first constructed by N. Wiener. He considered the case in which M is a Wiener process and h is a function of s alone. The above recipe works for constructing the Wiener integral, although Wiener used a different approach. The key is to observe that, for a step function h defined on the real line,

$$(1) \qquad E\left(\int_0^t h_s\,dW_s\right)^2 = \int_0^t h_s^2\,ds.$$

This equality guarantees sufficient continuity to define the integral of any square integrable function h with respect to W, the result being a square integrable random variable for each t. This construction is equivalent to the usual construction of integrals in the Hilbert space operator calculus of functional analysis.

Itô [14] realized that a random function h can be allowed in Wiener's integral if h is required to be adapted. In fact, in case h is a random adapted step function, equation (1) is true with only a slight modification—the right-hand side needs to be replaced by its expectation. Then Doob [9] convinced the world that the martingale property of the Wiener process was its key to success as a stochastic integrator when he proceeded to use a large class of martingales as integrators.

Exploiting a decomposition theorem of Meyer, often called the Meyer-Doob decomposition theorem, Kunita and Watanabe [17] provided a rather general theory of integration with respect to martingales. Since the general theory allows the integrator to have jumps, it is useful to place a condition stronger than adaptedness, called "predictability", on the integrand. A random process $A = (A(t,\omega))$ can be viewed as a function on $\mathbf{R}_+ \times \Omega$. The σ-algebra $\mathbf{P}$ of subsets of $\mathbf{R}_+ \times \Omega$ generated by left-continuous adapted random processes is called the class of predictable subsets of $\mathbf{R}_+ \times \Omega$ corresponding to $(\mathbf{F}_t)$. A random process is called predictable if it is $\mathbf{P}$ measurable. In this modern terminology, due to Doléans-Dade and Meyer [7], the Meyer-Doob decomposition theorem implies (under certain technical assumptions) that for any martingale M there is an increasing predictable random process $\langle M \rangle$ such that

$$(2) \qquad E\left(\int_0^t h_s\,dM_s\right)^2 = E\int_0^t h_s^2\,d\langle M\rangle_s$$

for predictable step functions h. This equality provides sufficient continuity to define the stochastic integral for predictable h such that the right-hand side of (2) is finite.

Concurrent with the development of stochastic integrals was the generalization of Itô's celebrated change of variable formula:

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds.$$

The last term on the right-hand side of Itô's formula reflects the fact that the Wiener process W is not absolutely continuous in t. Itô's formula is true for any twice continuously differentiable function f—in particular, no growth conditions are needed on f. However, a localization procedure in which processes are restricted to an ω-dependent interval $[0, \tau(\omega))$ is needed to overcome possible quick growth of $f'(W_s)$ when defining the dW integral. The resulting integral may not be a martingale, but is an example of what are now called local martingales.

If M is a martingale then Y defined by $Y_t = f(M_t)$ need not be a martingale or even a local martingale. Therefore Itô's formula must be extended to nonmartingale processes in order to obtain a class of processes which is closed under the operation $Z_t \to f(Z_t)$. A satisfactory extension was given by Doléans-Dade and Meyer [7] who established a generalization of Itô's formula in which W is replaced by any process Z which can be represented as $Z = M + V$, where M is a local martingale, V is an adapted process with sample paths of finite variation over bounded intervals, and the sample paths of both M and V are right continuous and have left limits at all t.

Processes Z with such a possibly nonunique representation have come to be called *semimartingales*. A consequence of the Doléans-Meyer formula is that if Z is a semimartingale then so is $Y = (f(Z_t))$ for twice continuously differentiable functions f.

Integrals with respect to a semimartingale Z were originally defined by integrating with respect to V and M separately and showing that the result does not depend on the choice of the representation $V + M$ for Z. However, such integrals can also be defined, in a manner closer to that of the martingale integral discussed above, by a method refined by Métivier and Pellaumail (see **[21]**), and used in the book under review. The key is the following fact. Given a semimartingale Z, there exists an increasing adapted right-continuous process C such that

$$E\left\{\left(\sup_{t\leq\tau}\int_0^t h_s\,dZ_s\right)^2\right\} \leq E\left\{C_{\tau^-}\int_0^{\tau^-} h_s^2\,dC_s\right\} \tag{3}$$

for all elementary predictable processes h and all stopping times τ. Métivier terms C a "control process" of Z ("control", here has nothing to do with stochastic control). General semimartingale integrals can be constructed and studied through direct use of (3). This suggests that the theory of stochastic integration should be extended to all integrands Z which admit control processes. Little is gained for processes in finite dimensions—a process (assumed adapted, right-continuous with left limits) admits a control process if and only if it is a semimartingale. However, the class of processes in infinite dimensions admitting a control process is strictly larger than the class of semimartingales.

The preceding gives a sense in which, for processes in finite dimensions, semimartingales are the most general stochastic integrators. A related sense is the following: If Z is a semimartingale, then, for finite t, μ_Z^t, defined by

$$\mu_Z^t(A) = \int_0^t 1_A\,dZ_s, \qquad A \in \mathbf{P},$$

is a bounded σ-additive L^0-valued measure on the σ-algebra $\mathbf{P}$ of predictable sets. Here L^0 denotes the space of random variables with the convergence in probability metric. Conversely, Dellacherie-Mokobodski-Meyer and K. Bichteler independently proved that the above characterizes semimartingales among all adapted, right-continuous-with-left-limits-random processes Z **[6, 21]**.

This fact indicates a close connection between stochastic integration and vector-valued measures.

The semimartingale calculus is a powerful tool for explicitly connecting the behavior of processes under two different probability measures to the Radon-Nikodym derivative of the measures. To see why, let P' be another probability distribution on the measure space $(\Omega, \mathbf{F})$ which is absolutely continuous with respect to P, let P_t' and P_t denote the respective restrictions of P' and P to $\mathbf{F}_t$, and let L_t denote the Radon-Nikodym derivative of P_t' with respect to P_t. Then L_t is a martingale. More generally, if M_t is a martingale with respect to P' then $M_t L_t$ is a martingale with respect to P. Any semimartingale on $(\Omega, \mathbf{F}, P)$ is also a semimartingale on $(\Omega, \mathbf{F}, P')$. Now the object of a typical dynamical estimation problem (known as a "filtering problem" to electrical engineers) is to compute an estimate for some possibly time-varying parameter from observations of the sample paths of some random process. Often estimates are continually undated as more observations become available. Different parameter values determine different probability distributions for the observed process, so that multiple probability distributions on the σ-algebras generated by the observed process naturally arise. These facts begin to explain the success of the semimartingale calculus in engineering statistical estimation contexts (consult [**20, 26**] for further information).

The next ingredient of semimartingale calculus is the stochastic integral (or formal differential) equation, a prototype of which was given by K. Itô [**15**] (see [**8, 13**] for the extension to semimartingale driving term):

$$(4) \qquad X_t = X_0 + \int_0^t m(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dW_s; \qquad X_0 \text{ given.}$$

Under Lipschitz hypothesis on m and σ, solutions to this equation are unique in the strong sense that distinct solutions agree for all t with probability one. Under much weaker conditions, one can still guarantee that the *distribution* of the solution is unique. For example, it is only necessary to require m to be bounded and measurable for appropriate σ [**13, 25**]. In a typical stochastic control problem, the function m, and possibly σ, in (4) depends on an additional variable u_s. The goal is to find a control $u = (u_s\colon 0 \leqslant s \leqslant T)$ which is a casual function of some "observed" random process related to X in order to maximize a reward such as $EJ(X_T)$. Uniqueness results with minimal regularity have been extremely useful in formulating stochastic control problems without making ad hoc smoothness assumptions on admissible controls [**3, 11, 12**].

The fact that $f(Z_t)$ yields a semimartingale if Z_t is a martingale makes it possible to study semimartingales with values in a differentiable manifold and to study stochastic differential equations in manifolds. Due to the second derivative term appearing in Itô's change of variable formula, second order tangent vectors are appropriate for specifying semimartingale trajectories [**22**]. Some regularity theorems—for example the surprising fact that the solution of the stochastic differential equation (4) for fixed t is continuously differentiable with respect to the initial value (if m and σ are smooth enough)—and stochastic calculus of variation results are most easily studied in the context of differential geometry. See [2, 22, 13] and the references therein for a good introduction to this area of current research.

It is possible to extend Itô's change of variable formula to situations when the second derivative of f only exists as a generalized function which is a finite signed measure. For example, if f is the difference of two convex functions and Z is a real-valued sample continuous semimartingale then

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s)\,ds + \int_{\mathbf{R}} f''(du) L_t^u(Z),$$

where $L_t^u(Z)$ is the "local-time of Z at u up to time t." (See [1] for an introduction and recent work using local time of semimartingales.) Recently local times have been shown to be an elegant and powerful tool in the study of uniqueness and comparison theorems for stochastic differential equations [18].

Two other areas of current interest involving semimartingale calculus are (1) the extension of the semimartingale calculus to multiple parameter random processes [19], and (2) the use of semimartingales to prove random process convergence theorems (see, e.g., [16]).

Some comments about Métivier's book are in order. Métivier's book and his earlier book written with Pellaumail [21] are the first to present stochastic integration using the notion of control processes described above. This is an appealing alternative to the original approach used in other texts. Also, the reader can grasp the essentials of *both* approaches using this book. Metivier's book (using [21] as a follow-up) is the best place to begin learning about stochastic integrals and differential equations for Banach-space-valued processes. Such processes, for example measure-valued processes, are gaining increased attention for biological and economic models.

Another notable feature of the book is the use of quasimartingales to present Doob's martingale inequalities in a novel fashion that is both elegant and well-matched to the study of stochastic integration presented later. A highlight is the application of martingale convergence concepts to the study of convergence of stochastic algorithms.

Some well-known application areas, such as nonlinear stochastic estimation and control, are not considered, and some applications that are given are far from convincing. For example, the "application" to queueing theory gives complicated expressions for modeling a rather simple queue using a semimartingale. The semimartingale calculus may be truly valuable in the analysis of queues, but the impression one gets (which is probably correct anyway) from Métivier is just the opposite. Another problem is that the book suffers from a high incidence of sometimes severe typographical errors.

The book is suitable for a text in a second year graduate special topics course, with measure theory and some measure theoretic probability as a prerequisite. The book also complements well the important works of Meyer and Dellacherie [5, 6], which are the most complete and elegantly written books on the semimartingale calculus and the "Strasbourg school" general theory of processes accompanying their study. However, [5 and 6] do not cover integration with respect to semimartingales in infinite dimensions, let alone integration with respect to the more general processes considered by Métivier.

Readers having difficulty tackling the general theory may find Chung's and William's more elementary book [4], or Elliot's book [10], a good place to start. Elliot's book follows [6] in organization but is less comprehensive and contains more examples and applications.

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