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THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SCHRÖDINGER EQUATIONS

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We consider the nonlinear Schrödinger equation with power interactions

$$\text{(NS)} \qquad i\partial u/\partial t = -\tfrac{1}{2}\Delta u + \lambda |u|^{p-1}u$$

in $\mathbf{R}^n$, $n \geq 2$, with $\lambda > 0$. Proposing a new method for studying the large time behavior of the solutions of (NS), we prove the following theorem. $H_0 = -\frac{1}{2}\Delta$ is the free Hamiltonian and

$$\Sigma = \{u \in L^2(\mathbf{R}^n); \|u\|_2 + \|\nabla u\|_2 + \|xu\|_2 < \infty\}, \tag{1}$$

where $\|u\|_q$ denotes the L^q-norm of u.

THEOREM. *Let $1 + 2/n < p < 1 + 4/(n-2)$. Then for any $u_0 \in \Sigma$ there exists a unique $u_\pm \in L^2(\mathbf{R}^n)$ such that the solution $u(t)$ of* (NS) *with $u(0) = u_0$ has the free asymptote $u_\pm$ as $t \to \pm\infty$:*

$$\lim_{t \to \pm\infty} \|u(t) - \exp(-itH_0)u_\pm\|_2 = 0. \tag{2}$$

REMARK. Since it is shown by Glassey [4] and Strauss [6] that if $1 < p \leq 1 + 2/n$ any nontrivial solution $u(t)$ of (NS) with $u(0) \in S$ never satisfies (2), our theorem achieves the least possible exponent $1 + 2/n$ for this direction.

In the sequel we shall prove the theorem. Our proof is based on the following observation: Since the asymptotic profile of the free evolution $\exp(-itH_0)f$ is given by $(1/it)^{n/2}\exp(ix^2/2t)\hat{f}(x/t)$ and (NS) is transformed by the conjugation C,

$$u(t,x) = (Cv)(t,x) = (1/it)^{n/2}\exp(ix^2/2t)\overline{v(1/t, x/t)}, \tag{3}$$

into the new equation

$$\text{(TNS)} \qquad i\partial v/\partial t = -\tfrac{1}{2}\Delta v + \lambda |t|^{n(p-1)/2-2}|v|^{p-1}v,$$

the relation (2) is equivalent to the existence of

$$\lim_{t \to \pm 0} v(t) \equiv v_\pm(0) \quad \text{in } L^2(\mathbf{R}^n). \tag{4}$$

Here and hereafter $\hat{f}$ and $\check{f}$ are the Fourier transform of f and the inverse Fourier transform of f, respectively. The equation (TNS) has almost the same form as (NS) and, for $p > 1 + 2/n$, $t^{n(p-1)/2-2}$ is integrable near $t = 0$. Thus we expect the existence of the limit (4) for those p's.

The equation (NS) has interested many authors and there is quite a body of literature. Among them, we mention the following which are related to

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our result. For $1 \le p < 1 + 4/(n-2)$, the global existence of the solution $u(t)$ of (NS) with $u(0) \in H^1(\mathbf{R}^n)$ is proved by Ginibre and Velo [**1**]. In [**2**] they also show the above theorem for $1 + 4/n < p < 1 + 4/(n-2)$ (see also Lin and Strauss [**5**]). The lower exponent $1 + 4/n$ is subsequently decreased to $\gamma(n) = (n + 2 + \sqrt{n^2 + 12n + 4})/2n$ in Strauss [**7**], but the allowed $u(0)$ are restricted to be small in a certain norm.

PROOF. From [**1**] and [**2**] we already know that (NS) has a unique global solution $u(t, \cdot) \in C(\mathbf{R}^1; \Sigma)$ with $u(0) = u_{\iota}$. We note that the solution of (NS) means the so-called mild solution of the integral equation associated with the differential equation (NS) (see [**1**]). Then a direct computation shows that $v(t) = (C^{-1}u)(t) \in C(\mathbf{R}^{\pm}; \Sigma)$ is a unique solution of (TNS).

We prove the theorem for $t \to +\infty$ with $1 + 2/n < p \le 1 + 4/n$ only. The other cases may be proved similarly. We first obtain two conservation laws for (TNS). We multiply (TNS) by $t^{2-n(p-1)/2}\partial\overline{v}/\partial t$ and take the real part. This leads us to

$$\begin{aligned} &t^{2-n(p-1)/2}\|\nabla v(t)\|_2^2 + \frac{4}{p+1}\int_{\mathbf{R}^n} |v(t,x)|^{p+1}\,dx \\ &\qquad \ge s^{2-n(p-1)/2}\|\nabla v(s)\|_2^2 + \frac{4}{p+1}\int_{\mathbf{R}^n} |v(s,x)|^{p+1}\,dx \end{aligned} \tag{5}$$

for all $0 < s \le t < +\infty$. We note that this rather formal calculation can be easily justified by the regularizing technique of Ginibre and Velo [**1**]. Next we multiply (TNS) by $\overline{v}$ and take the imaginary part to obtain

$$\|v(t)\|_2 = \|v(s)\|_2, \qquad 0 < s \le t < +\infty. \tag{6}$$

By (5) and (6) we conclude that

$$t^{2-n(p-1)/2}\|\nabla v(t)\|_2^2 < C_1, \quad \|v(t)\|_{p+1} < C_2, \quad \|v(t)\|_2 < C_3, \tag{7}$$

for all $t \in (0, 1]$, where C_1, C_2 and C_3 depend only on $\|v(1)\|_{H^1}$ and $\|v(1)\|_{p+1}$. Let $\varphi \in H^1(\mathbf{R}^n)$. By (TNS),

$$\begin{aligned} (v(t) - v(s), \varphi) &= \int_s^t \left(\frac{\partial v(\tau)}{\partial \tau}, \varphi\right) d\tau \\ &= -\frac{i}{2}\int_s^t (\nabla v(\tau), \nabla\varphi)\, t\tau \\ &\quad - i\int_s^t \tau^{n(p-1)/2-2}(|v(\tau)|^{p-1}v(\tau), \varphi)\, d\tau \end{aligned} \tag{8}$$

for $0 < t, s < +\infty$, where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbf{R}^n)$. Since $n(p-1)/2 - 2 > -1$ for $p > 1 + 2/n$ and $H^1(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n)$), (7) and (8) show that the weak limit

$$\operatorname*{w\text{-}lim}_{t \to +0} v(t) \equiv v(0) \tag{9}$$

exists in $L^2(\mathbf{R}^n)$. Now choose $\varphi = v(t)$ in (8). Then

$$\begin{aligned} |(v(t) - v(s), v(t))| &\le \frac{1}{2}\int_s^t \|\nabla v(\tau)\|_2\, d\tau \cdot \|\nabla v(t)\|_2 \\ &\quad + \int_s^t \tau^{n(p-1)/2-2}\|v(\tau)\|_{p+1}^p\, d\tau \cdot \|v(t)\|_{p+1}, \end{aligned} \tag{10}$$

for all $0 < s \le t < +\infty$. Applying (7) to (10), we have

$$|(v(t)-v(s),v(t))| \le C_4\left[\frac{4}{n(p-1)}\{t^{n(p-1)/2-1} - s^{n(p-1)/4}t^{n(p-1)/4-1}\} + \frac{2}{n(p-1)-2}\{t^{n(p-1)/2-1} - s^{n(p-1)/2-1}\}\right]. \tag{11}$$

Let $s \to +0$ and use (9) to obtain

$$|(v(t)-v(0),v(t))| \le C_5 t^{n(p-1)/2-1} \tag{12}$$

with $C_5 > 0$ depending only on n, p, $\|v(1)\|_{p+1}$ and $\|v(1)\|_{H^1}$. Therefore,

$$\begin{aligned} \|v(t)-v(0)\|_2^2 &= (v(t)-v(0),v(t)) - (v(t)-v(0),v(0)) \\ &\le C_5 t^{n(p-1)/2-1} + |(v(t)-v(0),v(0))| \\ &\to 0 \quad (t \to +0). \end{aligned} \tag{13}$$

Returning to (NS) we see that

$$\|\exp(-itH_0)\check{\bar{v}}(0) - u(t)\|_2 \to 0 \qquad (t \to +\infty), \tag{14}$$

as desired. □

The construction of wave operators and the asymptotic completeness problem will be discussed elsewhere.

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