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SINGULAR MINIMIZERS FOR REGULAR ONE-DIMENSIONAL PROBLEMS IN THE CALCULUS OF VARIATIONS

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We announce some surprising examples of regular one-dimensional problems of the calculus of variations possessing singular absolute minimizers. These minimizers *do not satisfy the usual integrated version of the Euler-Lagrange equation.* The examples all concern integrals of the form

$$I(u) = \int_a^b f(x, u(x), u'(x))\,dx,$$

where $[a, b]$ is a finite interval, $'$ denotes d/dx, $f = f(x, u, p)$ is C^∞, $f \geq 0$ and $f_{pp} > 0$ (regularity). We consider the problem of minimizing I in the set $\mathcal{A}$ of absolutely continuous functions $u\colon [a, b] \to \mathbf{R}$ satisfying $u(a) = \alpha$, $u(b) = \beta$, where α and β are given constants. As is well known, if a minimizer u of I in $\mathcal{A}$ is Lipschitz continuous then u is smooth and satisfies the Euler-Lagrange equation

$$\text{(EL)} \qquad (d/dx)f_p = f_u,$$

and the DuBois-Reymond equation

$$\text{(DBR)} \qquad (d/dx)(f - u'f_p) = f_x,$$

for all $x \in [a, b]$. A little-known partial regularity theorem of Tonelli [**1923**, Vol. 2, p. 359] asserts that given any minimizer u of I in $\mathcal{A}$, (i) there exists a closed set $E \subset [a, b]$ of measure zero such that $u \in C^\infty([a, b]\backslash E)$ and

$$\lim_{\substack{\operatorname{dist}(x,E)\to 0\\ x\notin E}} |u'(x)| = \infty,$$

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and (ii) for any $x \in [a,b]$ the limit

$$u'(x) = \lim_{\substack{\delta \to 0 \\ x+\delta \in [a,b]}} \frac{u(x+\delta) - u(x)}{\delta}$$

exists as an element of the extended real line $\overline{\mathbf{R}}$. In particular u satisfies (EL) and (DBR) on $[a,b]\backslash E$. As far as we are aware, our examples are the first showing that the Tonelli set E may be nonempty. (For a recent version of Tonelli's theorem for nonsmooth integrands see Clarke and Vinter [**1983**].)

Details of the proofs and further results will be published elsewhere.

EXAMPLE 1. *Minimize*

$$I(u) = \int_0^1 [(x^2 - u^3)^2 (u')^{14} + \epsilon (u')^2]\, dx$$

subject to $u(0) = 0$, $u(1) = k > 0$.

It is easily verified that if

$$0 < \epsilon < \epsilon_0 = \max_{t^3 \in [7/13,1]} (2t/3)^{12}(1-t^3)(13t^3 - 7) = .002474\ldots,$$

the corresponding Euler-Lagrange equation has an exact solution $u = kx^{2/3}$ on $(0,1]$ provided k has either of two values k_1, k_2 with $\frac{7}{13} < k_1^3 < k_2^3 < 1$. The underlying reason for the existence of these exact solutions is the scale invariance property

$$(*) \qquad f(\lambda x, \lambda^{2/3} u, \lambda^{-1/3} p) = \lambda^{-2/3} f(x,u,p), \qquad \lambda > 0,$$

of the integrand. This invariance is also responsible for the existence of transformations, namely $v = u^{3/2}$, $z = v/x$, $q = v'$ and $x = e^t$, converting (EL) into an *autonomous* first order system of ordinary differential equations in the first quadrant of the q,z plane. The critical points in $q > 0$, $z > 0$ of this system are precisely the points $q = z = k_1^{3/2}$ (a sink) and $q = z = k_2^{3/2}$ (a saddle). Furthermore, every smooth solution u of (EL) on $[0,1]$ with $u(0) = 0$, $u'(0) > 0$ corresponds to a single orbit $q(t)$, $z(t)$ leaving the origin with slope $3/2$. Provided $\epsilon > 0$ is sufficiently small it can be shown that this orbit is attracted as $t \to \infty$ to $q = z = k_1^{3/2}$. It follows that for $\epsilon > 0$ sufficiently small there exists $\delta = \delta(\epsilon) > 0$ such that if $k > k_2 - \delta$ there is no smooth solution of (EL) on $[0,1]$ satisfying the end conditions, and hence that I does not attain a minimum among Lipschitz functions. By applying the direct method of the calculus of variations and further analysis of the q,z phase portrait [in conjunction with a device due to Mania (cf. Example 2 below)], one concludes that for each $k \geq k_1 - \delta_1(\epsilon)$ there is a unique absolutely continuous minimizer u, such that $u \in C^\infty((0,1])$ and $u(x) \sim k_2 x^{2/3}$ as $x \to 0+$. (If $k = k_2$ then $u(x) = k_2 x^{2/3}$.) Hence the Tonelli set E consists of the single point $x = 0$. Since $u'(0) = \infty$ it follows easily that $f_u, f_x \notin L^1(0,1)$, so that u does not satisfy the integrated versions

$$\text{(IEL)} \qquad f_p = \int_0^x f_u + \text{constant},$$

$$\text{(IDBR)} \qquad f - u' f_p = \int_0^x f_x + \text{constant},$$

of (EL) and (DBR). If $k > 0$ is sufficiently small the minimizer is smooth and unique. If $k = k_1$ then, for all $0 < \epsilon < \epsilon_0$, $u(x) = k_1 x^{2/3}$ is *not* the minimizer.

EXAMPLE 2. *Minimize*

$$I(u) = \int_{-1}^{1} [(x^4 - u^6)^2 (u')^{2m} + \epsilon (u')^2]\, dx$$

subject to $u(-1) = -k$, $u(1) = k > 0$.

Here m is a positive integer. Note first that when $m = 13$ the integrand has the invariance property $(*)$, and that if $\epsilon > 0$ is sufficiently small there are two solutions, $u = k_1 |x|^{2/3} \operatorname{sign} x$, $u = k_2 |x|^{2/3} \operatorname{sign} x$, of (EL) for $x \neq 0$. For minimization problems such as in Example 1 the same phase portrait techniques are applicable. However, we now consider the case $m > 13$. We fix $k \in (0, 1]$ and let $\epsilon > 0$ be sufficiently small. Then I attains a minimum, and any minimizer u satisfies $u(0) = 0$, $u'(0) = +\infty$. Furthermore

$$\inf_{\substack{v \in W^{1,\infty}(-1,1) \\ v(\pm 1) = \pm k}} I(v) > I(u) \quad (\textit{the Lavrentiev phenomenon}).$$

Here the Tonelli set E contains at least one *interior* point, namely $x = 0$, and (IEL) and (IDBR) are not satisfied for any choice of lower limit in the integrals of f_u, f_x. These results are proved by adaptation of the argument of Mania [**1934**] (see also Cesari [**1983**, p. 514], who is responsible for the resurrection of the remarkable Lavrentiev phenomenon from the literature). The Lavrentiev phenomenon can be viewed as a kind of 'uncertainty principle': one cannot simultaneously approximate the minimizer u and minimum value m of I arbitrarily closely by a Lipschitz function.

EXAMPLE 3. We state this example as a proposition, since the explicit formula for the integrand is somewhat complicated.

PROPOSITION. *There exists a nonnegative C^∞ function $f = f(u,p)$ such that:*

(i) *for some $\alpha > 0$ the functional*

$$I(u) = \int_{-1}^{1} f(u, u')\, dx$$

has a unique minimizer $\overline{u}$ in the set $\mathcal{A}$ of absolutely continuous functions on $[-1, 1]$ satisfying $u(-1) = -\alpha$, $u(1) = \alpha$;

(ii) $\overline{u} \in C^\infty([-1,1]\backslash\{0\})$, $\overline{u}(0) = 0$, $\overline{u}'(0) = +\infty$, *and* $f_u(\overline{u}, \overline{u}') \notin L^1_{\text{loc}}(-1, 1)$, *so that* (IEL) *does not hold and* $E = \{0\}$;

(iii) $f_{pp} > 0$, $|p| \le f(u,p) \le \text{const}(1 + p^2)$ *for all* u, p, *and*

$$\lim_{|p| \to \infty} \frac{f(u,p)}{p} = \infty \quad \textit{for every } u \neq 0.$$

The proof proceeds by first establishing conclusions (i) and (ii) for the integral

$$J(u) = \int_{-1}^{1} (g'(u) u')^2\, dx,$$

where g is a suitable function satisfying $g'(u) > 0$ for $u \neq 0$, $g'(0) = 0$, by examining the values of $\overline{u}, \overline{u}'$, and finally constructing an appropriate integrand $f(u,p) \geq (g'(u)p)^2$ satisfying $f_{pp} > 0$ and such that $f(\overline{u},\overline{u}') = (g'(\overline{u})\overline{u}')^2$. Note that for integrands independent of x, (IDBR) always holds for a minimizer (cf. Tonelli [**1934**], Cesari [**1983**, p. 63]). If $f_{pp} > 0$, $f(u,p)/p \to \infty$ for all u then an argument based on (IDBR) shows that any minimizer is smooth and satisfies (EL) on $[-1,1]$, so that Example 3 is, in a sense, optimal.

It would be interesting to determine if analogues of Examples 1–3 hold for multiple integrals with integrands independent of u, such as those occurring in nonlinear elasticity, under growth hypotheses ensuring that any minimizer is continuous. If so, then the appearance of singularities in the gradient of u could be related to the onset of fracture. Finally, we remark that because of the Lavrentiev phenomenon care must be taken in the interpretation of minimizers obtained numerically via finite element schemes.

Most of the results concerning Examples 1 and 3 were obtained when Mizel was a U. K. Science and Engineering Research Council Visiting Fellow at Heriot-Watt University in 1981 and 1982. Further results were developed during a brief joint visit at the Institute for Mathematics and its Applications (University of Minnesota) and at the Mathematics Research Center (University of Wisconsin).

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