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ON THE RELATIONS BETWEEN CHARACTERISTIC CLASSES OF STABLE BUNDLES OF RANK 2 OVER AN ALGEBRAIC CURVE

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ABSTRACT. We describe a complete set of generators and relations for a certain quotient of the rational cohomology ring of the moduli space of stable bundles of rank 2 and fixed determinant of odd degree over a nonsingular complex algebraic curve. The formulae for the relations apply in any genus and are relatively simple.

1. Introduction. Let $S = U_L(2,1)$ denote the moduli space of stable bundles of rank 2 and determinant L of degree 1 over a nonsingular complete algebraic curve X of genus $g \geq 2$ defined over the complex numbers. The Betti numbers of S were determined some time ago in [**3**], and generators for $H^*(S;Q)$ were given in [**4**]. Recently there has been renewed interest in obtaining a complete description of $H^*(S;Q)$, particularly in connection with the work of M. F. Atiyah and R. Bott [**1**, §9]. David Mumford and Dave Bayer have performed some calculations on a computer, which provide evidence in support of some conjectures of Mumford. In this note we use the topological methods of [**3, 4**] to obtain some information about relations in $H^*(S;Q)$; these provide further support for Mumford's conjectures.

2. The main theorem. We recall the generators for $H^*(S;Q)$ given in [**4**], namely $\alpha \in H^2(S;Z)$; $\psi_1, \ldots, \psi_{2g} \in H^3(S;Z)$; $\beta \in H^4(S;Z)$. A little care is needed over the definition of the ψ_i. We first choose a *symplectic* basis $a_1, \ldots, a_{2g}$ for $H^1(X;Z)$ (with respect to the skew-symmetric form given by Poincaré duality); then the ψ_i are defined by the equation

$$\psi = \psi_1 \otimes a_1 + \cdots + \psi_{2g} \otimes a_{2g},$$

where ψ is the component in $H^3(S;Z) \otimes H^1(X;Z)$ of the second Chern class of a universal bundle on $S \times X$. We write

$$\sigma = \psi_1\psi_2 + \cdots + \psi_{2g-1}\psi_{2g} \in H^6(X;Z),$$

so that $\psi^2[X] = 2\sigma$.

THEOREM 1. *Let A denote the ring $H^*(S;Q)/\langle\beta\rangle$. Then the monomials*

(1) $$\alpha^s\psi_{q_1}\cdots\psi_{q_t} \quad (s,t \geq 0,\ 1 \leq q_1 < q_2 < \cdots < q_t \leq 2g,\ s+t < g)$$

form a basis for A as a vector space over Q. Moroever, whenever $s + t \geq g$,

(2) $$[\alpha^s + f_s(\alpha,\sigma)]\psi_{q_1}\cdots\psi_{q_t} = 0,$$

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where

$$\sum_{u\geq 1}\binom{s}{3u}\binom{3u-1}{2}\binom{3u-4}{2}\cdots\binom{2}{2}\alpha^{s-3u}(8\sigma)^u.$$

Note that (2) can be used to express every element of A as a linear combination of the basis elements (1); hence (1) is a complete set of relations for A as a graded Q-algebra. The elements α, β and σ correspond to those denoted by $h, h^2-4\nu$ and $-\theta$ in [**5**, §5]. It will be noted that our result, though it does not give a complete description of $H^*(S;Q)$, is a precise one; we hope that it will be of help in obtaining a complete result. Note further that $f_s(\alpha,\sigma)$ is independent of g; this provides partial verification of one of Mumford's conjectures.

3. Subsidiary results. We recall from [**3, 4**] the subspaces $S_0^{(g)}$ and $N^{(g)\prime}$ of $SU(2)^{2g}$ defined by

$$(A_1,\ldots,A_{2g})\in S_0^{(g)} \Leftrightarrow (A_1A_2A_1^{-1}A_2^1)\cdots(A_{2g-1}A_{2g}A_{2g-1}^{-1}A_{2g}^{-1}) = -I,$$

$$(A_1,\ldots,A_{2g})\in N^{(g)\prime} \Leftrightarrow \mathrm{Trace}[(A_1A_2A_1^{-1}A_2^1)\cdots(A_{2g-1}A_{2g}A_{2g-1}^{-1}A_{2g}^{-1})] \geq 0.$$

With the notation of [**4**, §3], and writing $\sigma = \mu_1\mu_2 + \cdots + \mu_{2g-1}\mu_{2g}$, we shall prove

THEOREM 2. *For $r\leq 3g-3$, the monomials*

$$(3)\qquad \lambda^s\mu_{q_1}\cdots\mu_{q_t} \qquad (s,t\geq 0,\ 1\leq q_1<q_2<\cdots<q_t\leq 2g,\ 2s+3t=r),$$

with $s+t\leq g-1$, form a basis for $H^r(S_0^{(g)};Q)$. Moreover, whenever $s+t\geq g$,

$$(4)\qquad [\lambda^s+g_s(\lambda,\sigma)]\mu_{q_1}\cdots\mu_{q_t}=0,$$

where

$$g_s(\lambda,\sigma)=f_s(\lambda,k\sigma/8)$$

and k is a constant independent of g.

THEOREM 3. *For $r\leq 3g$, the monomials* (3) *with $s+t\leq g$ form a basis of $H^r(N^{(g)\prime};Q)$. Moreover,* (4) *holds whenever $s+t\geq g+1$.*

4. Outline of proofs. Theorem 1 can be deduced from Theorem 2 by recalling [**2, 3**] that there is a principal $PU(2)$-fibration $p\colon S_0^{(g)}\to S$. Moreover, by [**4**, Proposition 2.6], the ideal $\langle\beta\rangle$ in $H^*(S;Q)$ coincides with the kernel of p^*, and it follows from the formulae for the Betti numbers in [**3**] that p^* is zero above degree $3g-3$. One can check also that $p^*(\psi_i)=\mu_i$: while $p^*(\alpha)$ is certainly a multiple of λ. So Theorem 2 implies a modified version of Theorem 1 in which (2) is replaced by $[\alpha^s+f_s(\alpha,k'\sigma)]\psi_{q_1}\cdots\psi_{q_t}=0$, where k' is a constant which could possibly depend on g. It remains to prove that $k'=1$, which we do by using the family of stable bundles constructed by Ramanan in [**5**, §4] and making some explicit computations.

The first assertion of Theorem 2 follows from [**4**, Proposition 3.4] by using the formulae of [**3**, Theorem 2]. The relations [**4**] follow from those of Theorem 3 by using the maps

$$h\colon N^{(g-1)\prime}\to S_0^{(g)} \qquad (\text{see [4, p. 341]}),$$

$$\rho\colon S_0^{(g)} \to S_0^{(g)} : \rho(A_1, \dots, A_{2g}) = (A_{2g-1}, A_{2g}, A_1, \dots, A_{2g-2}).$$

Let δ be an element of $H^r(S_0^{(g)}; Q)$ with $r \leq 3g-3$; by writing δ as a linear combination of (3) and making some direct computations, we see that

$$\delta = 0 \Leftrightarrow (\rho^i \circ h)^*(\delta) = 0 \quad \text{for } 0 \leq i \leq g-1. \tag{5}$$

A further direct computation proves (4).

Finally, the first assertion of Theorem 3 follows from [4, Proposition 3.3]. For the relations, we argue by induction on g, using the maps

$$l\colon N^{(g-1)'} \times N^{(1)'} \to N^{(g)'}, \qquad m\colon N^{(g-2)'} \times N^{(2)'} \to N^{(g)'}$$

defined in [4, p. 342; 3, p. 256], and an assertion similar to (5) with respect to these maps. The cases $g=1$ and $g=2$ must be checked separately. (For $g=2$ we get just one relation $\lambda^3 + k\sigma = 0$; this is the source of the constant k.)

Further details of the proofs, and some related results, will appear elsewhere.

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