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LOCAL RINGS OF FINITE SIMPLICIAL DIMENSION

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In this note R denotes a (noetherian, commutative) local ring with residue field k. Our purpose is to determine those R, over which k has finite (co-)homological dimension *as an R-algebra* in the simplicial theory of André [**1**] and Quillen [**11**]. Recall that regular local rings are characterized in this theory by the vanishing of the homology group $D_2(k|R)$. Furthermore, it is known that each of the conditions (i) $D_3(k|R) = 0$, (ii) $D_4(k|R) = 0$, (iii) $D_q(k|R) = 0$ for $q \geq 3$, is equivalent to R being a complete intersection, by which we mean that in some (hence in any) Cohen presentation of the completion $\hat{R}$ as a homomorphic image of a regular local ring $\tilde{R}$, the ideal $\operatorname{Ker}(\tilde{R} \to \hat{R})$ is generated by an $\tilde{R}$-regular sequence.

THEOREM 1. *If $D_q(k|R) = 0$ for q sufficiently large, then R is a local complete intersection.*

REMARK 1. The previous statement proves a conjecture of Quillen [**11**, Conjecture 11.7] and answers a question of André [**1**, p. 118]. When $\operatorname{char}(k) = 0$, its validity is established by [**11**, Theorem 7.3] and Gulliksen's result in [**10**].

REMARK 2. It has been shown by the author and Halperin [**4**] that in characteristic zero the conclusion of the theorem holds under the (much) weaker assumption that $D_q(k|R) = 0$ for *infinitely many* values of q. It is not known whether the restriction is essential, and in fact it is an open question, in any characteristic, whether the cotangent complex is rigid, i.e.: Does $D_q(k|R) = 0$ for *a single* $q \geq 1$ imply that R is a complete intersection?

The proof of Theorem 1 uses some precise information on the growth of the coefficients of the formal power series $P_R(t) = \sum_{j\geq 0} \dim_k \operatorname{Tor}_j^R(k,k)t^j$. For our present purpose it is best expressed in terms of the radius of convergence $r(P_R(t))$. Note that the inequality $r(P_R(t)) > 0$ has been known for a long time to hold for any local ring R, and that for complete intersections one even has $r(P_R(t)) \geq 1$.

THEOREM 2. *The inequality $r(P_R(t)) \geq 1$ characterizes complete intersections.*

REMARK 3. The last result has been conjectured both by Golod and by Gulliksen, and proved, in case $R = \bigoplus_{i\geq 0} R_i$ is graded with $R_0 = k$ a field of characteristic zero, by Felix and Thomas [**9**]. Results related to Theorem 2 are discussed in [**2**]; complete proofs will appear in [**3**].

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PROOF OF THEOREM 1. Denote by $\mathbf{L}_{k|R}$ the cotangent complex of the R-algebra k (so that $H_*(\mathbf{L}_{k|R}) = D_*(k|R)$ by definition), and by S^k the symmetric algebra functor, extended—dimensionwise—to simplicial k-vector spaces. There is a convergent "fundamental spectral sequence", due to Quillen [**11**, Theorem 6.3], such that

$$E_{p,q} = H_{p+q}(S^k_q \mathbf{L}_{k|R}) \Rightarrow \mathrm{Tor}^R_{p+q}(k,k).$$

With $E(t)$ denoting the formal power series $\sum_{j\geq 0}(\sum_{p+q=j} \dim_k E_{p,q})t^j$, this implies a coefficientwise inequality $E(t) \geq P_R(t)$; hence for the radii of convergence one obtains

$$r(P_R(t)) \geq r(E(t)). \tag{1}$$

The simplicial vector space $\mathbf{L}_{k|R}$ decomposes, according to Dold [**7**], in a direct sum $V \oplus (\bigoplus_i W_i)$ of simplicial vector spaces, such that $H_*(V) = 0$, $H_{n_i}(W_i) \simeq k$ for some integer n_i, and $H_j(W_i) = 0$ for $j \neq n_i$. Since, by the general results of [**1** and **11**], $D_q(k|R)$ is finite dimensional for each q, our assumption implies the direct sum above involves only finitely many spaces $W_1, \ldots, W_m$. By [**7**] again,

$$H_*(S^k\mathbf{L}_{k|R}) = H_*(S^kV) \otimes \bigotimes_{i=1}^m H_*(S^kW_i).$$

According to Dold and Thom [**8**], $H_*(S^kV) = k$, and

$$H_*(S^kW_i) \simeq H_*(K(\mathbf{Z}, n_i), k),$$

where $K(\mathbf{Z}, n)$ denotes, as always, the Eilenberg-Mac Lane space whose unique nontrivial homotopy group is infinite cyclic and located in degree n. Setting

$$\vartheta(n,k)(t) = \sum_{j\geq 0} \dim_k H_j(K(\mathbf{Z},n),k)t^j,$$

one can write

$$E(t) = \prod_{i=1}^m \vartheta(n_i,k)(t). \tag{2}$$

The circle S^1 being a familiar $K(\mathbf{Z},1)$, and the complex projective space $\mathbf{CP}^\infty$ being a $K(\mathbf{Z},2)$, one has, over any field k,

$$\vartheta(1,k) = (1+t), \qquad \vartheta(2,k) = (1-t^2)^{-1}. \tag{3}$$

Furthermore, the identity

$$\vartheta(n,k) = (1+(-1)^{n+1}t^n)^{(-1)^{n+1}} \tag{4}$$

is valid for all n (≥ 1), when $\mathrm{char}(k) = 0$.

Finally, when $\mathrm{char}(k) = p > 0$, one has

$$r(\vartheta(n,k)(t)) = 1 \quad \text{for } n \geq 3. \tag{5}$$

For $p = 2$ this is established by Serre as a consequence of his computation of the mod 2 cohomology of $K(\mathbf{Z}, n)$: cf. [**12**, §3, Theorem 5 and Corollary 1 to Theorem 2]. When p is odd, one can use in a similar way Cartan's isomorphism $H_*(K(\mathbf{Z},n),k) \simeq \Gamma_*(C_*)$, where Γ denotes the free algebra with divided powers, and C_* is a graded vector space, determined in [**6**]. More precisely, according to [**6**, Main Theorem and Theorem 3], $c_j = \dim C_j$ can be described as being the number of solutions of the equations

$$h_1 + 2\sum_{i=2}^{s} p^{i-1}h_i + 2\sum_{i=1}^{s-1} p^{i-1}u_i = j, \quad h_1 + 2\sum_{i=2}^{s} h_i + \sum_{i=1}^{s-1} u_i = n$$

in nonnegative integers s, h_i, u_i, subjected to the conditions $u_i \leq 1$; $h_i + u_i \geq 1$ for $i = 1, 2, \ldots, s-1$; $h_s \geq 1$. In particular, c_j does not exceed the number of decompositions of j as a sum of $2n-1$ nonnegative integers, hence

$$C(t) = \sum_{j\geq 0} c_j t^j \leq (1-t)^{-2n+1},$$

yielding the inequality $r(C(t)) \geq 1$. It can be replaced by an equality since $C(t)$ has integral coefficients and is not a polynomial. Because of this last circumstance one can also apply a result of Babenko [**5**], according to which $r(C(t)) = r(\vartheta(n,k)(t))$, hence (5) holds.

Formulas (1)–(5) show that Theorem 1 is a consequence of Theorem 2.

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