BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 10, Number 1, January 1984

ASYMPTOTIC ENUMERATION OF LATIN RECTANGLES

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A $k \times n$ *Latin rectangle* is a $k \times n$ matrix with entries from $\{1, 2, \ldots, n\}$ such that no entry occurs more than once in any row or column. (Thus each row is a permutation of the integers $1, 2, \ldots, n$.) Let $L(k, n)$ be the number of $k \times n$ Latin rectangles. An outstanding problem is to determine the asymptotic value of $L(k, n)$ as $n \to \infty$, with k bounded by a suitable function of n.

The first attack on this problem was made by Erdös and Kaplansky [1], who obtained the correct value for $k = O((\log n)^{3/2-\epsilon})$. The range of validity was later widened to $k = o(n^{1/3})$ by Yamamoto [8] and to $k = o(n^{1/2})$ by Stein [7]. We have obtained the correct value for $k = o(n^{6/7})$, and at the same time have sharpened the known approximations for fixed $k \geq 4$. Specifically, we have the following Theorem.

THEOREM. *Let $k = O(n^{1-\delta})$ for some fixed $\delta > 0$. Then*

$$L(k, n) = \frac{(n!)^{n+k}}{n^{nk}(n-k)!^n} \exp(k(k-1)l(k, n)),$$

where

$$l(k, n) = \frac{1}{4n} + \frac{k-1}{6n^2} + \frac{k^2-k-1}{8n^3} + \frac{12k^3 - 13k^2 - 13k - 6}{120n^4} + \frac{15k^4 - 18k^3 - 18k^2 - 28k + 47}{180n^5} + O\left(\frac{k^5}{n^6}\right).$$

The bound $l(k, n) \geq 0$ is a consequence of the van der Waerden permanent conjecture. It is interesting to note that the leading coefficients of the expansion of $l(k, n)$ are in harmonic progression. If this trend continues (which we cannot prove) it would suggest that

$$L(k, n) \sim \frac{(n!)^{n+k}}{n^{nk}(n-k)!^n}\left(1 - \frac{k}{n}\right)^{-n/2} e^{-k/2}$$

as $n \to \infty$ with $k = O(n^{1-\delta})$.

As with previous work, our Theorem is obtained by first estimating the average number of ways in which a $k \times n$ rectangle can be extended to a $(k+1) \times n$ rectangle by adding an extra row. The important new feature of our work is that is uses some of the recently developed theory concerning the matchings and rook polynomials [2–5].

Received by the editors March 30, 1983.
1980 *Mathematics Subject Classification*. Primary 05A15, 05B20.

From [3 or 5] we know that the number of ways of extending a given $k \times n$ rectangle R can be expressed as $\int_0^\infty e^{-x} r(x)\,dx$, where $r(x)$ is a polynomial of degree n determined by R. Now, all the zeros of $r(x)$ are known [4, Lemma 4.1 and Theorem 4.3] to lie in the real interval $[0, 4k-4)$. For $n \gg k$, this implies that the integrand is concentrated in a fairly small region near $x = n$. Moreover, the moments of the set of zeros of $r(x)$ enumerate a certain family of closed walks in a k-regular bipartite graph G associated with R [2]. By comparing these with another family of closed walks in G [6], we obtain an accurate estimate of the number of extensions of R in terms of the counts of certain small subgraphs (squares, etc.) in G. The average values of these counts are then estimated by another method and the Theorem follows.

A similar technique has been used to asymptotically enumerate disjoint perfect matchings in the complete graph. Details will appear elsewhere.

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