big Picard theorem and Schottky's theorem have function-theoretic extensions, as does Hadamard's three circles theorem. Analytic continuations and automorphic solutions are developed at the conclusion.

To summarize, the book is well organized and accurate. Its style is computational. The tables of contents, glossary of terms and references are detailed and timely. And, open questions are pointed out. The result is an informative reference that can be followed with interest.

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Finite groups and finite geometries, by T. Tsuzuku (translated from the 1976 Japanese version by A. Sevenster and T. Okuyama), Cambridge Tracts in Mathematics, Vol. 78, Cambridge University Press, 1982, x + 328 pp., $42.50. ISBN 0-5212-2242-7

All finite simple groups are now known.[1] This monumental classification project involved the efforts of numerous mathematicians and occupies many thousands of pages. Several of these group theorists are presently working hard to decrease the size of the proof. Nevertheless, it seems unlikely that the proof of this classification will become accessible to many mathematicians.

How should this classification be viewed by those not in group theory? It is clearly a remarkable result. But is it unapproachable? Is there any point in understanding parts of it? Can it be used outside of group theory, or is it just a marvelous technical feat designed only for internal consumption? It may even be tempting to ask: "What has this done for me lately?"

[1]Except that, as of this writing (April, 1983), the uniqueness of the Monster has not yet been established.

The search for *nontrivial* applications of the classification is still in its infancy. But there have already been some striking results obtained by using it in number theory, field theory, representation theory, computational complexity, model theory, finite geometry and combinatorics. For surveys, see [**3**, **4** and **7**]. (However, it is not at all surprising that these surveys are already slightly out of date.) More and more uses can be expected, both outside and inside group theory.

How much background is needed in order to apply the classification? This will clearly vary with the problem under consideration. In many instances, relatively little background is required. For example, only a very specific consequence of the classification may be needed, such as the correctness of Schreier's conjecture (the outer automorphism group of any finite simple group is solvable). Alternatively, only a precise list of all finite 2-transitive permutation groups may be needed, together with a couple of simple properties of each group on the list. Even if detailed information is needed for an application, only properties of the simple groups may be required: it may not be necessary to know anything at all about the intricacies of the proof of the classification. This all amounts to taking an engineering approach to using the classification: take its validity for granted (or regard it as a ridiculously technical axiom) and build upon it. (Of course, as with the engineer, it would be necessary to have faith in the underlying theory in order to believe that the resulting edifice would not crumble.)

Several applications described in [**3**, **4**, and **7**] do not even require any knowledge of the finite groups of Lie type. In many instances, much of the background can be found in textbooks. Moreover, several books have just appeared, making the background easier to acquire. Both [**2** and **8**] are primarily concerned with group theory and representation theory. The book under review blends group theory with finite geometry: group theory is seen as a subject which can, indeed, be applied elsewhere. A somewhat similar point of view can be found in [**1**]. Finally, there is also [**5**], which is concerned with a brief description of the proof of the classification and of the finite simple groups themselves.

In the future, books on finite group theory will certainly be somewhat influenced by the classification. Some topics are more significant than others for potential applications. Some previously important results may become trivial due to the classification. It may therefore seem as if books written more than a year or two ago have lost their purpose.[2] But this is certainly not the case. Techniques used in "outdated" theorems have already found post-classification uses. Moreover, basic material must be learned before the classification can be used. Finally, "outdated" theorems usually have more elegant and more informative proofs than a proof obtained by hurling the classification against a problem. (The word "outdated" is also suspect: many theorems which can now be called "outdated" were involved in the proof of the classification.)

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[2] This point of view was expressed to me by a representative of the publisher of the book under review.

Tsuzuku's book was first published in Japanese in 1976, before the end of the proof of the classification was in sight. It is primarily concerned with providing some of the more basic background for research involving permutation groups and for geometric research involving groups. The author is particularly interested in linking these two fields of research. Quoting from the preface: "The study of automorphism groups is... a powerful tool in the study of finite geometries. Given a finite geometry, the group of automorphisms is a finite group. Interesting geometric structures often give rise to interesting finite groups. Conversely, from the group theoretical point of view, there are many interesting problems in the areas related to finite geometries, such as the construction of new finite geometries and the determination of the finite geometry (sic) corresponding to a given finite group."

For example, consider the projective geometry $PG(n-1, q)$ determined by an n-dimensional vector space over a finite field with q elements, where $n \geqslant 3$. The group $PGL(n, q)$ acts 2-transitively on the points of $PG(n-1, q)$. Of course, it is not surprising that there is an intimate relation between $PG(n-1, q)$ and $PGL(n, q)$. The last two chapters of this book contain several characterizations of both $PG(n-1, q)$ and these 2-transitive permutation groups.

As another example, consider a group G and a subgroup H of G. Assume that H contains no nontrivial normal subgroup of G, and let X be the set of all right cosets of H in G. Then G acts transitively on X. Assume that it does not act 2-transitively, and let R be any G-orbit on $X \times X$ other than the diagonal $\{(x, x) \mid x \in X\}$. Then R determines a directed graph with vertex set X. Moreover, G acts transitively on both the vertices and edges of this directed graph. Since G is not 2-transitive, there are nonadjacent pairs of vertices, so the directed graph has some chance of providing some kind of information concerning G and H. Moreover, G is primitive on X (i.e., H is a maximal subgroup of G) if and only if this directed graph is connected for each choice of R. In particular, each primitive permutation group (other than 2-transitive ones) produces objects in this manner which can be studied from a nonalgebraic point of view.

The preceding examples indicate that finite geometries can be studied group theoretically and that permutation groups can be studied combinatorially. In fact, finite groups have become an indispensable tool in finite geometry. They arise most naturally when a geometry is assumed to be homogeneous. The most striking result of this sort is the Ostrom-Wagner theorem: A finite projective plane is isomorphic to some $PG(2, q)$ if and only if its automorphism group is 2-transitive on points.

On the other hand, geometric and combinatorial methods have had a significant impact on the classification. The most important example is Tits' use of buildings and polar spaces in the study of groups with a BN-pair [**9**]: Tits' characterization theorem, and variations on it, have been used extensively in the classification. The geometry of some of the sporadic simple groups (especially the Mathieu groups) has been essential to the construction and study of progressively larger and larger sporadic simple groups, culminating in the construction of the Monster [**6**]. However, both buildings and sporadic groups are too advanced for Tsuzuku's book.

The book starts from scratch (Lagrange's theorem; the usual isomorphism theorems; a dozen pages on linear algebra over arbitrary fields), and ends with theorems proved in the early 1970's. En route it covers many standard group theoretic topics: Sylow's theorem; the Schur-Zassenhaus theorem; an introduction to character theory, including Burnside's $p^a q^b$ theorem and Frobenius' theorem on the groups named after him; transitive, primitive and multiply transitive groups, including much of [**10**, Chapters 1–2]; and properties (such as the normal structure) of symmetric and alternating groups and of $\mathrm{PSL}(n, q)$. There are also short proofs of Thompson's theorem on the nilpotence of Frobenius kernels and Burnside's theorem on permutation groups of prime degree.

In other words, the book includes a standard, brief introduction to basic group theory. It also contains an introduction to finite geometry, which starts even before Sylow's theorem is stated and continues in later chapters, especially in the final two. Designs are introduced, with emphasis on those arising from projective and affine geometries. Directed graphs are constructed from primitive groups as indicated earlier. They are used to give a proof of a beautiful theorem of Sims concerning trivalent graphs, and to study other properties of suborbits. There is a long introduction to finite projective planes, culminating in proofs of the Ostrom-Wagner theorem and Wagner's generalization of it. Results on projective planes are used to give a short proof of the finite version of the Veblen and Young axiomatization of projective geometry —although Veblen and Young are never mentioned. Their result is then used to prove the Dembowski-Wagner Theorem and characterizations, due to O'Nan and Ito, of $\mathrm{PSL}(n, q)$ in its natural 2-transitive permutation representation. One of the strong points of this book is the inclusion of all of this fairly recent material.

Unfortunately, Tsuzuku's book is far too ambitious. It would be very difficult to use it as an introduction to the area, either at an elementary or a somewhat advanced level. Two books worth of material have been squeezed into 328 pages. Clearly, sacrifices had to be made. Most sections have at most one sentence containing any motivation. Theorems follow one after the other, with no lead-in comments. In particular, most results are proved without any indication of their importance either to mathematics or to later parts of the book. (Examples: The Artin-Zorn theorem, that finite alternative division rings are fields, appears very early—before the introduction to linear algebra—with no indication of its relevance to the book's subject matter. The importance of Thompson's theorem on Frobenius kernels is mentioned in an Epilogue on p. 317, but not when the theorem is proved on p. 144. The significance of Sims' trivalent graph theorem is never mentioned. Burnside's theorem on permutation groups of prime degree is called "classic", but there is not even one sentence explaining why until p. 318 in the Epilogue.) Of course, it is easy for the reader to deduce importance from the number of pages taken up by a proof, but that does not quite seem adequate.

Additional space is saved by not providing any exercises. Still more space is saved by streamlining proofs. Tsuzuku has provided many nice proofs. Statements such as "it is easy to show that" almost always are, indeed, easy. On the other hand, many parts of a proof that would normally be regarded as lemmas are frequently packed into page-long paragraphs with proofs in parentheses. Very long paragraphs are also often used to introduce terminology and to prove several properties of the newly defined objects. (Examples: On pp. 106–107 there is a $1\frac{1}{2}$-page paragraph containing the definition of, and elementary properties of, second cohomology groups. On pp. 124–125 there is a 1-page paragraph concerning properties of submodules.) Probably the clearest instance of this technique is the last paragraph of §1.2.2, on pp. 21–22, in which the following are introduced in turn: π-group, p-group, Hall π-subgroup, Sylow p-subgroup, π-complement, $O_\pi(G)$, $O_{\pi_1,\ldots,\pi_r}(G)$, $O^\pi(G)$ and $O^{\pi_1,\ldots,\pi_r}(G)$. (Sylow's theorem is first stated and proved much later on pp. 85–86.)

Finally, few examples are provided. Alternating groups are defined on p. 148. $\mathrm{PSL}(n, q)$ is studied on pp. 226–237. No other examples are described. Moreover, there isn't a single example of a Sylow subgroup of nonprime order. Similarly, there is no example of a finite projective plane not of the form $\mathrm{PG}(2, q)$—nor even a statement that many such planes have been constructed.

All of this condensation gives this book the appearance of lecture notes. It is not a textbook. It would be difficult for a student to learn from it. The book gives the impression that a well-chosen collection of basic techniques and theorems should be sufficient for a student. I find it hard to imagine that elementary or slightly advanced material can be learned without examples and motivation. A short Epilogue contains a few historical remarks, suggests further reading, mentions sporadic groups and Suzuki groups, and contains a definition of the symplectic group $\mathrm{PSp}(2n, q)$. All of this is too brief and too late.

The quality of the translation can be discerned from the portion quoted earlier. There are almost no mathematical or typographical errors. Almost all definitions and terminology are completely standard (with the exception of the definition of $\mathrm{SL}(n, K)$ on p. 55, and of projective and affine designs on pp. 75, 77).

For me, the nicest part of Tsuzuku's presentation involves one of the older results in the book. I know of no comparably accessible proof of any version of the Veblen and Young axiomatization. This fundamental result is quoted often, in a large number of research papers, but few proofs are in print. (Admittedly, Tsuzuku only handles the finite case, but that is the one used most often.) This and other parts of the book suggest that it would be a very good introduction to its subject if there were an additional 30 or 40 pages containing exercises, many more examples, and much more motivation.

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Lagrangian analysis and quantum mechanics, a mathematical structure related to asymptotic expansions and the Maslov index, by Jean Leray, the MIT Press, Cambridge, Mass., 1982, xvii + 271 pp., $35.00. ISBN 0-2621-2087-9

Semi-classical approximation in quantum mechanics, by V. P. Maslov and M. V. Fedoriuk, Mathematical Physics and Applied Mathematics, vol. 7, D. Reidel Publishing Company, Dordrecht:Holland/Boston:U.S.A./London:England, 1981, ix + 294 pp., Cloth Dfl. 125.00/U.S. $66.00. ISBN 9-0277-1219-0

1. It is a fundamental principle in quantum mechanics that, when the time and distance scales in a system are large enough relative to Planck's constant h, the system will approximately obey the laws of classical, Newtonian mechanics. To confound the uninitiated who think that physical constants are immutable this is usually rephrased: in the limit $h \to 0$ quantum mechanics tends to Newtonian mechanics. In either form this principle says very little. Quantum mechanics would not be widely accepted if it did not predict that boulders and freight trains obey Newton's laws. Quasi-classical approximations express this limiting behavior in more useful ways, through formulas for expectations, energy levels, etc. which are asymptotic to the exact formulas as $h \to 0$. The paradigm of such a formula is Bohr's energy quantization law. Bohr actually deduced this *before* the "exact" formula was introduced by Schrödinger. Nonetheless, Bohr's law can be rederived and generalized as a quasi-classical approximation. Quasi-classical approximations for problems with more than one degree of freedom are rather new. The first book to deal with them in some generality was Maslov's remarkable monograph [**8**]. In his preface to the French translation of [**8**] Jean Leray noted that a mathematician reading it would read much more between the lines than on them. *Quasi-classical approximations in quantum mechanics* (= QAQM) and *Lagrangian analysis and quantum mechanics* (= LAQM) are not so much sequels to [**8**] as systematic efforts to fill in those missing lines. (In this review we use the exact English translation of the Russian title of the book of Maslov and Fedoriuk. "Quasi-classical" and "semi-classical" appear to be used equally often in the English literature.)