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# CHARACTERIZATION OF STRICTLY CONVEX DOMAINS BIHOLOMORPHIC TO A CIRCULAR DOMAIN

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Recently parabolic exhaustions have been used successfully to classify complex spaces (Stoll [**6** and **7**], Burns [**2**], P. Wong [**9**]). Here we define parabolic exhaustions for strictly convex domains in $\mathbf{C}^n$ and give necessary and sufficient conditions for such a domain to be biholomorphically equivalent to a circular domain or even to the ball.

The results presented in this note are part of the Ph.D. Thesis the author is completing under the direction of Professor W. Stoll. Details and more implications will appear at a later date.

A strictly convex domain $D \subset \mathbf{C}^n$ is a domain for which there exists a defining function whose real Hessian is strictly positive on $T_x(\partial D)$ for all $x \in \partial D$. Let $\Delta$ be the open unit disk in $\mathbf{C}$ and let $S$ be the unit sphere in $\mathbf{C}^n$. Let $D \subset \mathbf{C}^n$ be a strictly convex domain and $p \in D$ be any point. For $b \in S$ Lempert [**4**] constructs an extremal map $F(\square, b)\colon \Delta \to D$ which is holomorphic with $F(0,b) = p$ and $F'(0,b) = \lambda$ where $1/\lambda > 0$ is the length of $b$ in the infinitesimal Kobayashi metric of $D$ at $p$. These conditions determine $F(\square, b)$ uniquely and the map extends smoothly to an embedding $F(\square, b)\colon \overline{\Delta} \to \overline{D}$. Also $F\colon \overline{\Delta} \times S \to \overline{D}$ is of class $C^\infty$ and surjective. One and only one function $\tau\colon \overline{D} \to \mathbf{R}_+$ exists such that $\tau(F(z,b)) = |z|^2$ for $(z,b) \in \overline{\Delta} \times S$. Then $\tau$ is a continuous exhaustion of $\overline{D}$, positive and of class $C^\infty$ on $\overline{D} \setminus \{p\}$. Also $\tau \equiv 1$ on $\partial D$. We refer to $\tau$ as the *Lempert exhaustion* of $D$ at $p$.

THEOREM 1. *The Lempert exhaustion at $p$ is strictly parabolic on $D_* = D \setminus \{p\}$, which means that on $D_*$*

(1) $dd^c\tau > 0$,

(2) $dd^c \log \tau \geq 0$,

(3) $(dd^c \log \tau)^n \equiv 0$.

Properties (2) and (3) were proved by Lempert. For each $r$ with $0 < r < 1$ the pseudoball $D(r) = \{x \in D \mid \tau(x) < r^2\}$ is a ball in the Kobayashi distance. By an argument of harmonic functions, it is shown that the Hessian of the defining function $\tau - r^2$ of $D(r)$ is positive on the tangent space of $\partial D(r)$ at every point of $\partial D(r)$. Hence $D(r)$ is strictly convex and $dd^c\tau > 0$ follows easily.

The strictly parabolic function $\tau$ defines a Monge-Ampère foliation on $D$ which coincides with the foliation defined by $F$. More precisely if

$$X = X^\mu \frac{\partial}{\partial z^\mu} = \tau_{\overline{\nu}} \tau^{\overline{\nu}\mu} \frac{\partial}{\partial z^\mu}$$

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is the complex gradient vector field associated to $\tau$ (see Stoll [6]), then we have

PROPOSITION 2. *If* $b \in S$ *and* $0 \neq z \in \Delta$, *then* $X(F(z,b)) = zF'(z,b)$.

Since $F'(0,b) \neq 0$ for all $b \in S$, a smooth, complete circular domain $G$ is given by

$$G = \{zb \mid z \in \mathbf{C}, b \in S, |z| < \|F'(0,b)\|\}.$$

A homeomorphism $h\colon \overline{G} \to \overline{D}$, called the *circular representation* of $D$ at $p$, is defined by

$$h(zc) = F\left(z, \frac{c}{\|c\|}\right)$$

for all $z \in \overline{\Delta}$ and $c \in \partial G$. The map $h$ is of class $C^\infty$ on $G_* = G \setminus \{0\}$ with $h(0) = p$. It is holomorphic along each disk through the origin and if $M$ is the Minkowski functional of $G$, then $\tau \circ h = M^2$.

PROPOSITION 3. *The map* $h\colon G \to D$ *is biholomorphic iff* $h$ *is of class* $C^\infty$ *at* 0.

PROPOSITION 4. *Let* $D_j$ *be strictly convex domains for* $j = 1, 2$. *Let* $h_j\colon G_j \to D_j$ *be the circular representation of* $D_j$ *at* $p_j \in D_j$. *If* $\phi\colon D_1 \to D_2$ *is a biholomorphic map with* $\phi(p_1) = p_2$, *then* $\psi = h_2^{-1} \circ \phi \circ h_1\colon G_1 \to G_2$ *is biholomorphic and in fact is the* $\mathbf{C}$*-linear map* $d\phi(p_1)$ *restricted to* $G_1$.

Thus the isotropy group of $\mathrm{Aut}(D)$ at $p$ lifts to a group of linear automorphisms of $G$.

The Monge-Ampère foliation is holomorphic if and only if $X$ is a holomorphic vector field.

THEOREM 5. *The Monge-Ampère foliation associated to the Lempert exhaustion* $\tau$ *is holomorphic iff the circular representation map is biholomorphic.*

It is most interesting to find conditions which force the Monge-Ampère foliation to be holomorphic. A one parameter group $\alpha\colon \mathbf{R} \times D \to D$ of automorphisms of $D$ is said to be *uniform* at $p \in D$ if $\alpha(t,p) = p$ for all $t \in \mathbf{R}$, $d\dot{\alpha}(0,p)\varsigma = \lambda\varsigma$ for all $\varsigma \in \mathbf{C}^n$ and some constant $\lambda \in \mathbf{C}$. Here $\dot{\alpha} = \partial\alpha/\partial t$. If $\alpha$ is so given, let $Y$ be the associated vector field, $Y(q) = \dot{\alpha}(0,q)$ for $q \in D$. Then $\alpha$ lifts to a one parameter group of automorphisms of $G$ uniform at 0. Also the vector field $X$ lifts to $\tilde{X} = z^\mu \partial/\partial z^\mu$ which enables us to show that $X = cY$ for some constant $c$. Since $Y$ is holomorphic, so is $X$. By Theorem 5 we conclude

THEOREM 6. *$D$ is biholomorphic to a circular domain iff there exists a one parameter group of automorphisms of $D$ uniform at some point of $D$.*

This theorem generalizes a classical result in two variables of H. Cartan [**3**].

We say that $D$ is *rotational* at $p \in D$ iff there is a one parameter group of automorphisms of $D$ uniform at $p$. Thus a strictly convex domain is biholomorphic to a circular domain iff it is rotational at least at one point. Combining results of Braun, Kaup and Upmeier [**1**], Rosay [**5**] and B. Wong [**8**], we obtain

THEOREM 7. *A strictly convex domain is biholomorphic to the ball iff it is rotational at least at two points.*

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