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## STEENROD-SITNIKOV HOMOLOGY FOR ARBITRARY SPACES

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**1. Introduction.** In order to establish an Alexander duality theorem for compact subsets of $S^n$, N. E. Steenrod introduced in 1940 a new type of homology of metric compacta. The same problem led K. A. Sitnikov in 1951 to an equivalent theory. In 1960 J. Milnor [**7**] gave an axiomatic characterization of the Steenrod-Sitnikov homology. Several authors extended the theory to the case of Hausdorff compact spaces (see, e.g., [**8**, **9**, **7** and **1**]).

The purpose of this announcement is to define a Steenrod-Sitnikov homology theory for arbitrary topological spaces. We refer to it as strong homology. It is obtained by first developing a strong homology of inverse systems. The transition from spaces to systems is achieved by means of ANR-resolutions, a new tool developed by S. Mardešić in [**5**] (also see [**6**]). Strong homology groups of a space are then defined as strong homology groups of any one of its ANR-resolutions. It is a consequence of our approach that strong homology is actually a functor on the strong shape category SSh introduced in [**4**].

**2. Strong homology of inverse systems.** We consider only inverse systems of topological spaces and maps $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ over directed cofinite sets. By a map of systems $f\colon \mathbf{X} \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ we mean an increasing function $\varphi\colon M \to \Lambda$ and a collection of maps $f_\mu\colon X_{\varphi(\mu)} \to Y_\mu$, $\mu \in M$, satisfying

$$f_\mu p_{\varphi(\mu)\varphi(\mu')} = q_{\mu\mu'} f_{\mu'}, \qquad \mu \le \mu'. \tag{1}$$

For a fixed Abelian group $G$ we associate with $\mathbf{X}$ a chain complex $C_\#(\mathbf{X}; G)$, defined as follows. Let $\Lambda^n$, $n \ge 0$, denote the set of all increasing sequences $\lambda = (\lambda_0, \dots, \lambda_n)$ from $\Lambda$. A strong $p$-chain of $\mathbf{X}$, $p \ge 0$, is a function $x$, which assigns to every $\lambda \in \Lambda^n$ a singular $(p+n)$-chain $x_\lambda \in C_{p+n}(X_{\lambda_0}; G)$. The boundary operator $d\colon C_{p+1}(\mathbf{X}; G) \to C_p(\mathbf{X}; G)$ is defined by the formula

$$(-1)^n (dx)_\lambda = \partial(x_\lambda) - p_{\lambda_0\lambda_1\#} x_{\lambda_0} - \sum_{j=1}^{n} (-1)^j x_{\lambda_j}; \tag{2}$$

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here $\partial$ denotes the boundary of singular chains and $\lambda_j$ is obtained from $\lambda$ by omitting $\lambda_j$. By definition, $H_p(\mathbf{X};G) = H_p(C_\#(\mathbf{X};G))$. A map $f\colon \mathbf{X} \to \mathbf{Y}$ induces a chain mapping $f_\#\colon C_\#(\mathbf{X};G) \to C_\#(\mathbf{Y};G)$ defined by

$$(3) \qquad (f_\# x)_{\boldsymbol{\mu}} = f_{\mu_0\#}(x_{\varphi(\mu_0)\dots\varphi(\mu_n)}), \qquad \boldsymbol{\mu} = (\mu_0,\dots,\mu_n) \in M^n.$$

One proves that $(gf)_\# = g_\# f_\#$. Consequently, $f$ induces a functorial homomorphism $f_*\colon H_p(\mathbf{X};G) \to H_p(\mathbf{Y};G)$.

**3. Coherent prohomotopy.** Extending and simplifying previous work of Lisica [**2, 3**], the authors have defined in [**4**] a coherent prohomotopy category CPHTop. Its objects are systems $\mathbf{X}$ as in §2. The morphisms are coherent homotopy classes of coherent maps of systems $f\colon \mathbf{X} \to \mathbf{Y}$, defined as follows. $f$ consists of an increasing function $\varphi\colon M \to \Lambda$ and of maps

$$f_{\boldsymbol{\mu}}\colon \Delta^n \times X_{\varphi(\mu_n)} \to Y_{\mu_0}, \qquad \boldsymbol{\mu} = (\mu_0,\dots,\mu_n) \in M^n, \qquad n \geq 0,$$

which satisfy

$$(4) \qquad f_{\boldsymbol{\mu}}(\partial_j^n t, x) = \begin{cases} q_{\mu_0\mu_1} f_{\boldsymbol{\mu}_0}(t,x), & j = 0, \\ f_{\boldsymbol{\mu}_j}(t,x), & 0 < j < n, \\ f_{\boldsymbol{\mu}_n}(t, p_{\varphi(\mu_{n-1})\varphi(\mu_n)}(x)), & j = n, \end{cases}$$

$$(5) \qquad f_{\boldsymbol{\mu}}(\sigma_j^n t, x) = f_{\boldsymbol{\mu}^j}(t,x), \qquad 0 \leq j \leq n;$$

here $\partial_j^n\colon \Delta^{n-1} \to \Delta^n$, $\sigma_j^n\colon \Delta^{n+1} \to \Delta^n$ are the usual face and degeneracy operators and $\boldsymbol{\mu}_j$ ($\boldsymbol{\mu}^j$) is obtained from $\boldsymbol{\mu}$ by omitting (repeating) $\mu_j$. Every map of systems can be viewed as a coherent map by putting $f_{\boldsymbol{\mu}}(t,x) = f_{\mu_0} p_{\varphi(\mu_0)\varphi(\mu_n)}(x)$. A coherent homotopy from $f$ to $f'$ is a coherent map $F\colon I \times \mathbf{X} \to \mathbf{Y}$, given by $\Phi \geq \varphi, \varphi'$ and $F_{\boldsymbol{\mu}}$ such that

$$(6) \quad F(t,0,x) = f_{\boldsymbol{\mu}}(t, p_{\varphi(\mu_n)\Phi(\mu_n)}(x)), \qquad F(t,1,x) = f'_{\boldsymbol{\mu}}(t, p_{\varphi'(\mu_n)\Phi(\mu_n)}(x)).$$

To define composition $fg$ of $f$ and $g\colon \mathbf{Y} \to \mathbf{Z} = (Z_\nu, \pi_{\nu\nu'}, N)$, one decomposes $\Delta^n$ into subpolyhedra

$$P_i^n\{(t_0,\dots,t_n) \in \Delta^n : t_0 + \cdots + t_{i-1} \leq \tfrac{1}{2} \leq t_0 + \cdots + t_i\}, \qquad i = 0,\dots,n,$$

and considers maps $\alpha_i^n\colon P_i^n \to \Delta^{n-i}$, $\beta_i^n\colon P_i^n \to \Delta^i$, where $\alpha_i^n(t) = (\#, 2t_{i+1},\dots,2t_n)$, $\beta_i^n(t) = (2t_0,\dots,2t_{i-1},\#)$, $\# = 1-$sum of remaining terms. Then

$$(7) \quad (gf)_{\nu_0\cdots\nu_n}(t,x) = g_{\nu_0\cdots\nu_i}(\beta_i^n(t), f_{\psi(\nu_i)\cdots\psi(\nu_n)}(\alpha_i^n(t),x)), \qquad t \in P_i^n.$$

With every coherent map $f\colon \mathbf{X} \to \mathbf{Y}$ we now associate a chain mapping $f_\#\colon C_\#(\mathbf{X};G) \to C_\#(\mathbf{Y};G)$, given by

$$(8) \quad (f_\# x)_{\boldsymbol{\mu}} = \sum_{i=0}^{n} f_{\mu_0\cdots\mu_i\#}(\Delta^i \times x_{\varphi(\mu_i)\cdots\varphi(\mu_n)}), \qquad \boldsymbol{\mu} \in M^n, x \in C_p(\mathbf{X};G).$$

If $f$ is a map of systems, then (3) and (8) give chain homotopic chain maps. Chain maps $(gf)_\#$ and $g_\# f_\#$ are chain homotopic. Coherently homotopic coherent maps induce chain homotopic chain maps. Consequently, strong homology is a functor of CPHTop. The proof of these assertions requires a tedious verification of explicit formulas giving the desired chain homotopies.

**4. Resolutions.** Let $p\colon X \to \mathbf{X}$ be a map of systems, i.e. a collection of maps $p_\lambda\colon X \to X_\lambda$ such that $p_{\lambda\lambda'}p_{\lambda'} = p_\lambda$ for $\lambda \le \lambda'$. The map $p$ is called a resolution of the space $X$ provided the following conditions hold for any ANR (for metric spaces) $P$ and any open covering $\mathcal{V}$ of $P$:

(R1) Every map $f\colon X \to P$ admits a $\lambda \in \Lambda$ and a map $g\colon X_\lambda \to P$ such that the maps $f$ and $gp_\lambda$ are $\mathcal{V}$-near.

(R2) There exists an open covering $\mathcal{V}'$ of $P$ such that whenever $\lambda \in \Lambda$ and maps $g, g'\colon X_\lambda \to P$ have the property that $gp_\lambda$ and $g'p_\lambda$ are $\mathcal{V}'$-near, then there exists a $\lambda' \ge \lambda$ such that $gp_{\lambda\lambda'}$ and $g'p_{\lambda\lambda'}$ are $\mathcal{V}$-near maps.

The resolution of a map $f\colon X \to Y$ consists of resolutions $p\colon X \to \mathbf{X}$, $\mathbf{q}\colon Y \to \mathbf{Y}$ and of a map of systems $g\colon \mathbf{X} \to \mathbf{Y}$ such that $gp = qf$.

It was proved in [**5**] that topological spaces and maps always have ANR-resolutions (all $X_\lambda$ and $Y_\mu$ are ANR's).

The following factorization theorem is crucial for the construction of our theory. Let $p\colon X \to \mathbf{X}$ be a resolution of $X$, let $\mathbf{Y}$ be an inverse system of ANR's and let $f\colon X \to \mathbf{Y}$ be a coherent map. Then there exists a coherent map $g\colon \mathbf{X} \to \mathbf{Y}$ such that $f$ and $gp$ are coherently homotopic. Moreover, $g$ is unique up to coherent homotopy.

The proof of this theorem is rather long. It involves construction of the maps $g_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in M^n$, by induction on $n$, using essentially cofiniteness of $M$, face and degeneracy properties of coherent maps and the uniqueness of linear homotopies in convex sets (see [**4**]).

**5. Strong homology of spaces.** $H_p(X;G)$ is defined as $H_p(\mathbf{X};G)$, where $p\colon X \to \mathbf{X}$ is an ANR-resolution of $\mathbf{X}$. The homomorphism $f_*\colon H_p(X;G) \to H_p(Y;G)$ induced by a map can be defined as $g_*\colon H_p(\mathbf{X};G) \to H_p(Y;G)$ (see §2), where $(p,q,g)$ is an ANR-resolution of $f$. More generally, if we have only ANR-resolutions $p\colon X \to \mathbf{X}$ and $q\colon Y \to \mathbf{Y}$ of $X$ and $Y$, we apply to $f$ the factorization theorem and obtain a coherent map $g\colon \mathbf{X} \to \mathbf{Y}$, which induces $g_*$ as in §3.

In [**4**] the authors defined a strong shape category SSh whose objects are all topological spaces. Morphisms $F\colon X \to Y$ are given by triples $(p,q,g)$, where $p$, $q$ are ANR-resolutions and $g$ is a morphism of CPHTop. If we assign to $F$ the homomorphism $g_*$, we see that strong homology is actually a functor on SSh. In particular, it satisfies the homotopy axiom. For ANR's and CW-complexes, strong and singular homologies coincide.

All our results also hold for pairs $(X,A)$. The obtained homology is exact whenever $A$ is $\mathcal{P}$-embedded in $X$, e.g., when $X$ is paracompact and $A$ is closed. Restricted to compact metric pairs the theory satisfies the Milnor axioms.

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