RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

SOME RESULTS IN HARMONIC ANALYSIS IN $\mathbf{R}^n$, FOR $n \to \infty$

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1. Introduction. The purpose of this note is to bring to light some further results whose thrust is that certain fundamental estimates in harmonic analysis in $\mathbf{R}^n$ have formulations with bounds independent of n, as $n \to \infty$.

It was shown previously (see [**3**, and **4**]) that for the basic maximal function M, defined by

$$Mf(x) = \sup_B \frac{1}{m(B)} \int_B |f(x-y)|\, dy$$

(with the sup taken over all balls B centered at the origin), one has

$$\|M(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p \leq \infty, \tag{1}$$

with bound A_p independent of n. Here we show that the analogue holds for the basic singular integrals, the "Riesz transforms" and their powers. While such results may have some interest on their own (the usual proofs give bounds which increase rapidly as $n \to \infty$), their validity raises certain further general questions and leads to speculation which we shall indulge in briefly at the end of this note.

2. The theorem. In $\mathbf{R}^n$ we define the familiar Riesz transforms by $(R_j f)\hat{}\,(\xi) = i(\xi_j/|\xi|)\hat{f}(\xi)$, $j = 1, \ldots, n$, and write $R = (R_1, \ldots, R_n)$; also $|R(f)(x)|$ will stand for $(\sum_{j=1}^n |R_j(f)(x)|^2)^{1/2}$.

THEOREM.

$$\|R(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p < \infty,$$

with A_p independent of n.

Received by the editors January 25, 1983.
1980 *Mathematics Subject Classification*. 42B25, 52B20.

The proof does not follow the usual argument for proving estimates for singular integrals, but instead in analogy with (1), it ultimately relies on square functions. The precise square functions that are relevant here are the "g-functions" (e.g. their closely related variants "Lusin functions" would not do). Thus we use

$$g(f) = \left(\int_0^\infty |\nabla P^t f|^2 t\, dt \right) \quad \text{and} \quad g_1(f) = \left(\int_0^\infty \left| \frac{\partial P^t f}{\partial t} \right|^2 t\, dt \right)^{1/2},$$

where $t \to P^t$ is the semigroup of Poisson integrals, and

$$|\nabla P^t f|^2 = \left| \frac{\partial P^t f}{\partial t} \right|^2 + \sum_{j=1}^n \left| \frac{\partial P^t f}{\partial x_j} \right|.$$

We now sketch the proof of the theorem. One observes first that

$$g_1(Rf)(x) \leq g(f)(x), \tag{2}$$

because

$$\frac{\partial P^t}{\partial t}(R_j f) = \frac{\partial}{\partial x_j}(P^t f).$$

Next one shows, whenever $f \geq 0$, that

$$\|g(f)\|_p \leq A_p \|f\|_p, \qquad 1 < p \leq 2, \tag{3}$$

with A_p independent of n.

In fact by the argument of [**2**, pp. 86–88] one has $\Delta(P^t f)^p = p(p-1)(P^t f)^{p-2} |\nabla P^t f|^2$; also $\|\sup_{t>0} P^t f\|_p \leq A'_p \|f\|_p$, by (1); and $\int_{\mathbf{R}^n} I(x)\, dx = \|f\|_p^p$, with $I(x) = \int_0^\infty t \Delta (P^t f)^p(x)\, dt$. Finally

$$(g(f)(x))^2 \leq \frac{1}{p(p-1)} \sup_{t>0} (P^t(f)(x))^{2-p} I(x).$$

The last step is to prove that

$$\|f\|_p \leq B_p \|g_1(f)\|_p, \qquad 1 < p \leq 2,$$

B_p independent of n, and here we must assume that f takes its values in a (d-dimensional) Hilbert space. For it we use a different approach—the theory of symmetric diffusion semigroups. We observe that the semigroup P^t satisfies all the required axioms (I–IV, [**1**, p. 65]). We can then invoke the Corollary 2 on p. 120, if we notice that in the present case the projection E_0 is zero. However a reexamination of the proof shows that the bound is independent of n (because the dimension n does not enter in the general axioms), and the results are equally valid in the Hilbert space version we need, and so B_p is also independent of d.

Once (4) is proved, the theorem follows from (2) and (3).

3. Further results and remarks. (i) Results analogous to (1) and the theorem hold also for:

(α) The "higher Riesz" transforms.

(β) For fractional integration of imaginary order, i.e. the boundedness on L^p, $1 < p < \infty$, of $(-\Delta)^{i\gamma}$, γ real, with bounds independent of the dimension n.

(γ) Inequalities in $\mathbf{R}^n$ of the form

$$\|\nabla f\|_r \leq A_{p,q}(\|\Delta f\|_p + \|f\|_q)$$

with $1 < p,q < \infty$, $1/r = \frac{1}{2}(1/p + 1/q)$, with $A_{p,q}$ independent of n.

(δ) The spherical maximal function, $\mathcal{M}$, with

$$(\mathcal{M}f)(x) = \sup_{t>0} \int_{|y|=1} |f(x-ty)|\, d\sigma(y),$$

and $d\sigma$ the normalized uniform measure on the unit sphere in $\mathbf{R}^n$. The assertion is $\|\mathcal{M}(f)\|_p \leq A_p\|f\|_p$, with $n > \max(p/(p-1), 2)$, and A_p independent of n.

(ii) The above results raise the following general question. Can one find an appropriate infinite-dimensional formulation of (that part of) harmonic analysis in $\mathbf{R}^n$, which displays in a natural way the above uniformity in n? A related question is to study the "limit as $n \to \infty$" of the above results, insofar as such limits may have a meaning. One might guess that a further understanding of these questions would involve, among other things, notions from probability theory: i.e. Brownian motion and possibly some variant of the central limit theorem.

References

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