expository in character, and they are quite good. I think that Chapter 0 gives a good introduction to the basic facts about representations. Chapter 1 discusses the case of $SL_2(\mathbf{R})$ and can be recommended as a very readable description of the infinite-dimensional representations of $SL_2(\mathbf{R})$. It needs few prerequisites. This cannot be said of the book as a whole, though. The subject matter of the book is difficult, and has many ramifications. This imposes high requirements on a prospective reader. To start with, he needs a thorough familiarity with complex Lie algebras and their representations.

The author makes an effort to present things clearly and efficiently, and usually succeeds in achieving this.

It is to be expected that future developments of the theory expounded in Vogan's book will lead to improvements and simplifications. I hope that the book will stimulate readers to find such improvements. Their efforts will be well spent.

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An introduction to homological algebra, by Joseph J. Rotman,[1] Academic Press, New York, 1979, xi + 376 pp., $26.50.

Homological algebra was invented by Henri Cartan and Samuel Eilenberg after World War II. It is essentially a technique borrowed from topology and

[1]The author writes that a complete list of errata for the first printing is available from the Educational Department of Academic Press, New York. All these errors have been corrected in the second printing.

applied to module theory. Inasmuch as it has a subject matter, this may be described as the deviation from exactness of the tensor product and the Hom-functor. I shall attempt to make this description more precise.

Given an associative ring R with identity, let A_R, B_R and C_R be right R-modules, ${}_RD$ a left R-module, and consider the additive functors

$$\mathrm{Hom}_R(A,-):\mathrm{Mod}\,R \to \mathbf{Ab}, \qquad \mathrm{Hom}_R(-,B):(\mathrm{Mod}\,R)^{\mathrm{op}} \to \mathbf{Ab},$$
$$C\otimes_R -: R\,\mathrm{Mod} \to \mathbf{Ab}, \qquad -\otimes_R D:\mathrm{Mod}\,R \to \mathbf{Ab}.$$

The first preserves products and kernels, and so does the second (more precisely, it sends coproducts and cokernels of Mod R to products and kernels of **Ab** respectively), while the third preserves coproducts and cokernels, as does the fourth. Actually, we need not worry about the fourth functor, as it is just the mirror image of the third and we did not mention the mirror images of the first and second.

To be *exact* an additive functor has to preserve both kernels and cokernels. When are the above functors exact?

P_R is called *projective* if $\mathrm{Hom}_R(P,-)$ is exact,
I_R is called *injective* if $\mathrm{Hom}_R(-,I)$ is exact,
F_R is called *flat* if $F\otimes_R -$ is exact.

These important concepts and their properties constitute what may be called elementary homological algebra. They have revitalized ring theory and commutative algebra, justifying many old notions and motivating several fertile new ones. Thus, *semisimple* rings are rings all of whose modules are projective (or injective) and *von Neumann regular* rings are rings all of whose modules are flat. *Dedekind* domains are those integral domains for which all divisible modules are injective and *Prüfer* domains are those for which all finitely generated torsionfree modules are projective. Among the important new notions that have turned up in ring theory are *right perfect* rings, all of whose flat right modules are projective, and *right hereditary* rings, for which all submodules of projective right modules are projective (or factor modules of injective right modules are injective). Moreover, there is an intimate connection between rings of quotients and injective hulls. Look at any modern text on ring theory and you will find that concepts inspired by elementary homological algebra predominate.

So much about the elementary aspects of homological algebra. To explain its more advanced aspects a small amount of machinery is necessary. Consider a sequence of modules and homomorphisms:

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots \qquad (n\in\mathbf{Z}).$$

This is called a *complex* if $d_nd_{n+1}=0$; that is, $\mathrm{im}\,d_{n+1}\subseteq \ker d_n$, for all integers n. It is called an *exact* sequence if $\mathrm{im}\,d_{n+1}=\ker d_n$ for all n.

It is perhaps unnecessary to remind the reader that the exactness of $0\to A\to B$, $B\to C\to 0$ and $0\to A\to B\to C\to 0$ means that $A\to B$ is a monomorphism, $B\to C$ is an epimorphism and $C\cong B/A'$ with $A'\cong A$ respectively.

For a complex $\mathbf{A}$ one defines the *homology* modules

$$H_n(\mathbf{A}) = \ker d_n / \operatorname{im} d_{n+1}.$$

Given any *short* exact sequence of complexes

$$\mathbf{0} \to \mathbf{A}' \to \mathbf{A} \to \mathbf{A}'' \to \mathbf{0},$$

there is a so-called *connecting* homomorphism

$$\partial_n : H_n(\mathbf{A}'') \to H_{n-1}(\mathbf{A}'),$$

giving rise to a *long* exact sequence

$$\cdots \to H_n(\mathbf{A}) \xrightarrow{p_n} H_n(\mathbf{A}'') \xrightarrow{\partial_n} H_{n-1}(\mathbf{A}') \xrightarrow{i_{n-1}} H_{n-1}(\mathbf{A}) \to \cdots.$$

Given any module $M = M_0$, there is a free module F_0 and an epimorphism $F_0 \to M_0$, hence an exact sequence

$$0 \to M_1 \to F_0 \to M_0 \to 0.$$

Repeating the same argument for M_1, and so on, we obtain an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where the F_n are free modules. Such an exact sequence is called a *free resolution* of M. *Projective* and *flat resolutions* are defined similarly; they exist because

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{flat}.$$

An *injective resolution* of M is an exact sequence

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots,$$

where the I^n are injective modules. It exists because every module can be embedded in an injective one. This can be seen in various ways; my favourite is the following, probably due to Northcott. Let $M^* = \operatorname{Hom}_R(M, \mathbf{Q}/\mathbf{Z})$; then M^* is a right module if M is a left module (or the other way around). Composing the canonical monomorphism $M \to M^{**}$ with the monomorphism $M^{**} \to F^*$ obtained from the epimorphism $F \to M^*$ where F is free, we obtain a monomorphism $M \to F^*$ where F^* is injective.

Consider now projective, injective and flat resolutions of A, B and C as follows:

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0,$$
$$0 \to B \to I^0 \to I^1 \to I^2 \to \cdots,$$
$$\cdots \to F_2 \to F_1 \to F_0 \to C \to 0,$$

and form the three complexes

$$0 \to \operatorname{Hom}_R(P_0, B) \to \operatorname{Hom}_R(P_1, B) \to \cdots,$$
$$0 \to \operatorname{Hom}_R(A, I^0) \to \operatorname{Hom}_R(A, I^1) \to \cdots,$$
$$\cdots \to F_1 \otimes_R D \to F_0 \otimes_R D \to 0.$$

For any natural number n, H_n of the first complex is called $\operatorname{Ext}^n_R(A, B)$. Surprisingly it agrees with H_n of the second complex. It follows that $\operatorname{Ext}^n_R(A, B)$ is independent of the particular choice of the projective resolution of A or of the injective resolution of B. H_{-n} of the third complex is called $\mathrm{Tor}_n^R(C, D)$. It could also have been obtained from a flat resolution of D. Thus $\mathrm{Tor}_n^R(C, D)$ is independent of the choice of either flat resolution.

What have these constructions to do with the description of homological algebra at the beginning of this review? Take for instance the second complex. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ of modules, one easily obtains a short exact sequence of complexes, hence a long exact sequence of Abelian groups as follows:

$$0 \to \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B) \to \mathrm{Ext}^1(A'', B) \to \cdots.$$

This shows how, in a manner of speaking, $\mathrm{Ext}^1(-, B)$ measures the deviation from exactness of the functor $\mathrm{Hom}(-, B)$. The situations regarding $\mathrm{Hom}(A, -)$ and $C \otimes -$ are quite similar.

Why the names Ext and Tor? $\mathrm{Ext}^1(A, B)$ may be regarded as the set of *extensions* of B by A, that is, the set of short exact sequences $0 \to B \to C \to A \to 0$ modulo an obvious equivalence relation. If R is a domain with quotient field Q, then $\mathrm{Tor}_1(Q/R, D)$ is the torsion submodule of D. However, the second etymology is somewhat misleading, as the true nature of torsion submodules is only captured by torsion theories, a more recent subject, albeit less prestigious than homological algebra.

Advanced homological algebra may also be applied to ring theory through the notion of dimension. One says the *projective dimension* of a module A is $\leq n$ provided $\mathrm{Ext}^{n+1}(A, -)$ is the zero functor. Injective and flat dimensions are defined similarly. The *left global dimension* of a ring R is the supremum of all projective (equivalently of injective) dimensions of left R-modules.

Although several books on homological algebra have appeared since its inception, the present text has much to recommend it, both contentwise and pedagogically. All the matters discussed above and much more are set forth admirably. The reader is gently taken by the hand and led to master the abstract machinery in easy steps, always bearing the concrete applications in mind.

One may quarrel with the author in some details of exposition. He defines Ext twice using the same notation before showing that the two definitions agree, relenting only in the next chapter by introducing an alternative notation for one of the two notions. He defines Tor from projective resolutions instead of the more general and more natural flat ones, basing himself on the dubious principle that covariant functors require projective resolutions, and only points out later that flat resolutions could have been used instead.

This book is particularly strong in applications. Here are some of the highlights. After a tour of modern ring theory, the author discusses the Quillen-Suslin theorem answering Serre's conjecture: if R is a polynomial ring over a field, every finitely generated projective module is free. He presents Vaserstein's version of Suslin's proof and sketches Quillen's proof.

He establishes Hilbert's theorem on syzygies in the form: if k is a field, the global dimension of $k[t_1, \ldots, t_n]$ is n. He gives an outline of the proof of the theorem of Auslander-Buchsbaum-Nagata, which asserts that every commutative Noetherian local ring with finite global dimension is a unique factorization domain. (But where is it shown that such a ring is a domain?)

Two chapters are devoted to group extensions. The cohomology and homology groups of the group G with coefficients in the $\mathbf{Z}G$-module A are defined by

$$H^n(G, A) = \mathrm{Ext}^n_{\mathbf{Z}G}(\mathbf{Z}, A), \qquad H_n(G, A) = \mathrm{Tor}^{\mathbf{Z}G}_n(\mathbf{Z}, A),$$

and the lower-dimensional ones are interpreted. Of particular interest are $H^2(G, A)$, the group of extensions of A by G, and $H_2(G, A)$, the Schur multiplier of G according to a formula by Hopf.

There is a final chapter on spectral sequences, which occupies about one sixth of the book and which emphasizes their use as a technique for computing homology. The reviewer admits regretfully that he did not read this chapter.

It is unusual for textbook writers to bother much about who originated what. The present author should therefore be applauded for going out of his way to attribute credit for theorems and proofs.

I found this book pleasant and stimulating reading. Its enthusiastic style is infectious and managed to rekindle my interest in the subject.

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Schottky groups and Mumford curves, by L. Gerritzen and M. van der Put, Lecture Notes in Math., vol. 817, Springer-Verlag, Berlin and New York, 1980, viii + 316 pp., $19.54.

I suspect that even in this modern age of categories and functors, most mathematicians still view p-adic analysis with a certain amount of disdain. If you're one such person, before you flip back to the research announcements, please allow me to show you a very important result, due to John Tate, that will change your mind, I hope.

We begin by reviewing some well-known concepts. Let H be the upper half-plane $\{x + iy \in \mathbf{C} \mid y > 0\}$. Let $\tau \in H$ and let $L = \{\mathbf{Z} + \mathbf{Z}\tau\}$. Thus, L is a two-dimensional lattice and it is a standard beautiful fact that $\mathbf{C}/L = E$ "is" an elliptic curve. This may be seen in either of the following ways:

(a) The field of L-invariant meromorphic functions on $\mathbf{C}$ forms an elliptic function field; or,

(b) the Riemann Surface E may be embedded into $\mathbf{P}^2(\mathbf{C})$ as a nonsingular cubic

$$y^2 = 4x^3 - g_2 x - g_3,$$

via the use of the classical Weierstrass $\wp$-function: In this last case we set, as functions of E,

$$\Delta = g_2^3 - 27 g_3^2 \quad \text{and} \quad j = (12 g_2)^3/\Delta.$$

It is well known that $\Delta \neq 0$ and that j characterizes E up to isomorphism (algebraic or complex, they are equivalent). We can, in fact, study such cubics over any algebraically closed field in a similar manner.