BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

Q VALUED FUNCTIONS MINIMIZING DIRICHLET'S INTEGRAL AND THE REGULARITY OF AREA MINIMIZING RECTIFIABLE CURRENTS UP TO CODIMENSION TWO

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We announce several results of an extensive study [**A**] of the size of singular sets in oriented m dimensional surfaces which are area minimizing in $m+l$ dimensional Riemannian manifolds. Our principal result is that the Hausdorff dimension of such a singular set does not exceed $m-2$. Examples show this is the best possible such general estimate when $l \geq 2$, i.e., when branching singularities are possible. The general existence of such surfaces of least area is well known in a variety of settings [**F**, 5.1.6].

In order to obtain estimates on branching of area minimizing surfaces we were led to use Taylor's expansion in terms of first derivatives at 0 to approximate the nonparametric area integrand by Dirichlet's integrand. Accordingly, we study branched coverings of regions in $\mathbf{R}^m$ which are graphs of multiple valued functions minimizing the integral of Dirichlet's integrand. As a central estimate in our analysis of area minimizing surfaces we show that the Hausdorff dimension of the branch set of such a minimizing covering does not exceed $m-2$.

To state several results in more detail we use the terminology of [**F**]. Suppose that A is a bounded open subset of $\mathbf{R}^m$ with smooth boundary, and let k, l, m, n, Q be positive integers with $k \geq 3$, $l \leq n$, and $m \geq 2$.

INTERIOR REGULARITY OF ORIENTED AREA MINIMIZING SURFACES. *Suppose N is an $m+l$ dimensional submanifold of $\mathbf{R}^{m+n}$ of class $k+2$ and that T is an m dimensional rectifiable current in $\mathbf{R}^{m+n}$ which is absolutely area minimizing with respect to N. Then there is an open subset U of $\mathbf{R}^{m+n}$ such that $\operatorname{spt} T \cap U$ is an m dimensional minimal submanifold of N of class k and the Hausdorff dimension of $\operatorname{spt} T \sim (U \cup \operatorname{spt} \partial T)$ does not exceed $m-2$.*

For such area minimizing T we have additionally

SINGULARITY STRATIFICATION BY TANGENT CONE TYPE. *Whenever $p \in \operatorname{spt} T \sim \operatorname{spt} \partial T$ and S is an oriented tangent cone to T at p then*

$$P(S) = \mathbf{R}^{m+n} \cap \{x : \theta^m(\|S\|, x) = \theta^m(\|S\|, 0) = \theta^m(\|T\|, p)\}$$

Received by the editors November 11, 1982.

1980 *Mathematics Subject Classification*. Primary 45F22; Secondary 53A10.

[1]This work was supported in part by grants from the National Science Foundation and from the Institute for Advanced Study in Princeton.

0273-0979/82/0000-1203/$01.50

is either the point $\{0\}$ *or a linear subspace of* $\mathbf{R}^{m+n}$ *with* $m-1 \neq \dim P(S) \leq m$. *Furthermore, for each* $j \in \{0, 1, \ldots, m-2, m\}$, *the Hausdorff dimension of*

$$(\operatorname{spt} T \sim \operatorname{spt} \partial T) \cap \{p : j = \sup\{\dim P(S) \colon S \textit{ is an oriented tangent cone to } T \textit{ at } p\}\}$$

does not exceed j.

We denote by $\mathbf{Q}$ the space of all 0 dimensional integral currents V in $\mathbf{R}^n$ for which $Q = \mathbf{M}(V) = \langle V, 1 \rangle$ with metric given by setting

$$\begin{aligned} &\operatorname{dist}([\![p(1)]\!] + \cdots + [\![p(Q)]\!], [\![q(1)]\!] + \cdots + [\![q(Q)]\!]) \\ &= \inf\left\{ \left(\sum_{i=1}^{Q} |p(i) - q(\sigma(i))|^2 \right)^{1/2} : \sigma \text{ is a permutation of } \{1, \ldots, Q\} \right\} \end{aligned}$$

whenever $p(1), \ldots, p(Q)$, $q(1), \ldots, q(Q) \in \mathbf{R}^n$. For Lipschitz $\mathbf{Q}$ valued functions we show a Lipschitz extension theorem analogous to Kirszbraun's theorem, an almost everywhere Q fold affine approximation theorem analogous to Rademacher's theorem, and also show that each Lipschitz function $A \to \mathbf{Q}$ induces a natural chain mapping of degree 0 from the chain complex of real flat chains having supports in A into the chain complex of real flat chains in $\mathbf{R}^n$. In terms of Dirichlet's integral naturally defined for appropriate functions $A \to \mathbf{Q}$ we have the following central results.

EXISTENCE AND REGULARITY OF DIRICHLET INTEGRAL MINIMIZING $\mathbf{Q}$ VALUED FUNCTIONS. *For each appropriate function* $g \colon \partial A \to \mathbf{Q}$ *there exists a (strictly defined but not necessarily unique) function* $f \colon A \to \mathbf{Q}$ *having boundary values* g *and of least Dirichlet integral among such functions. Furthermore, each such minimizing* f *is Hölder continuous, and* $A \times \mathbf{R}^n \cap \{(x, y) : y \in \operatorname{spt}(f(x))\}$ *is an* m *dimensional real analytic (harmonic) submanifold of* $A \times \mathbf{R}^n$ *except possibly for a closed set of Hausdorff dimension not exceeding* $m-2$.

Assuming that m and n and even integers and the usual complex identifications have been made, we show that the $\mathbf{Q}$ valued function produced by projection mapping slicing of a complex holomorphic chain in $A \times \mathbf{R}^n$ associated with a Q fold analytic branched covering of A is uniquely Dirichlet integral minimizing. Our Hausdorff codimension two singularity estimate for Dirichlet integral minimizing $\mathbf{Q}$ valued functions is thus the best possible.

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