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ON THE VANISHING OF POINCARÉ SERIES OF RATIONAL FUNCTIONS

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1. Let Γ be a finitely generated nonelementary Kleinian group with region of discontinuity Ω and limit set Λ. Let $\lambda(z)|dz|$ be the Poincaré metric on Ω (normalized to have constant negative curvature -1). Let $q \in \mathbf{Z}$, $q \geq 2$. A *cusp form for* Γ *of weight* $(-2q)$ is a holomorphic function φ on Ω satisfying

$$\varphi(\gamma z)\gamma'(z)^q = \varphi(z), \quad \text{for all } \gamma \in \Gamma, \text{ for all } z \in \Omega, \tag{1}$$

and either (hence both) of the following equivalent conditions:

$$\iint_{\Omega/\Gamma} \lambda(z)^{2-q}|\varphi(z)\, dz \wedge d\bar{z}| < \infty; \tag{2}$$

$$\sup_{z\in\Omega}\{\lambda(z)^{-q}|\varphi(z)|\} < \infty. \tag{3}$$

The equivalence of (2) and (3) shows that the *Peterson scalar product*

$$\langle\varphi, \psi\rangle = i\iint_{\Omega/\Gamma} \lambda(z)^{2-2q}\varphi(z)\overline{\psi(z)}\, dz \wedge \overline{dz} \tag{4}$$

induces a Hilbert space structure on the space of cusp forms.

Let Δ be a Γ-invariant union of components of Ω, and define $\mathbf{A}_q(\Delta)$ to be the space of cusp forms for Γ of weight $(-2q)$ that vanish on $\Omega\backslash\Delta$. Abbreviate $\mathbf{A}_q(\Omega)$ by $\mathbf{A}_q$.[2]

Define R_q to be the space of rational functions f such that
(5) f is holomorphic on Ω,
(6) f has only simple poles (on Λ), and

$$\begin{aligned} f(z) &= O(|z|^{-2q}), && z \to \infty \text{ if } \infty \in \Omega, \text{ and} \\ f(z) &= O(|z|^{-(2q-1)}), && z \to \infty \text{ if } \infty \in \Lambda. \end{aligned} \tag{7}$$

If $f \in R_q$, then the *Poincaré series*

$$\sum_{\gamma\in\Gamma} f(\gamma z)\gamma'(z)^q, \qquad z \in \Omega, \tag{8}$$

converges absolutely and uniformly on compact subsets of Ω and defines a cusp form $\Theta_q f \in \mathbf{A}_q$. Bers [**3**] has shown that

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[2]The group Γ is fixed throughout this paper. We hence suppress in the notation the dependence on Γ of the various spaces and operators considered.

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$$\Theta_q : R_q \to \mathbf{A}_q$$

is a surjective linear operator. The starting point of this investigation was the following theorem that quantitatively strengthens Bers' result.

THEOREM 1. *Let $a_1, \ldots, a_{2q-1}$ be $(2q-1)$ distinct points in Λ, and let $\gamma_1, \ldots, \gamma_N$ generate Γ (define $\gamma_0 = I$). Then $\Theta_q | R_q^0$ is surjective, where $R_q^0 = \{f \in R_q | f$ is holomorphic except possibly at $\gamma_j(a_k)$, $k = 1, \ldots, 2q-1$, $j = 0, \ldots, N\}$.*[3]

In certain cases $\Theta_q | R_q^0$ is an isomorphism. Spanning sets for Γ Fuchsian were obtained by Hejhal [4]. For $q = 2$, and Γ Fuchsian, Wolpert [**11**] obtained bases, as did Kra and Maskit [**7**] for Γ geometrically finite function groups.

2. We turn now to the more interesting *vanishing problem* raised by Poincaré [**10**, p. 249] (see also Petersson [**9**] and Hejhal [**4**]. Find necessary and sufficient conditions for $\Theta_q f$ to vanish identically on Ω (or Δ) for $f \in R_q$.

For $\psi \in \mathbf{A}_q(\Delta)$, the unique *Bers potential* $F = F_\psi$ for the *canonical generalized Beltrami coefficient* $\mu = \lambda^{2-2q}\overline{\psi}$ that vanishes at a_k, $k = 1, \ldots, 2q-1$, is given by

$$(9) \quad F(z) = \frac{(z-a_1)\cdots(z-a_{2q-1})}{2\pi i} \iint_\Omega \frac{\mu(\varsigma)\, d\varsigma \wedge d\overline{\varsigma}}{(\varsigma - z)(\varsigma - a)\cdots(\varsigma - a_{2q-1})}, \qquad z \in \mathbf{C}.$$

For $z \in \Lambda \backslash \{a_1, \ldots, a_{2q-1}\}$, we have (see Kra [**5**, Chapter V])

$$(10) \qquad F_\psi(z) = \langle \varphi(z, \cdot), \psi \rangle,$$

where

$$(11) \qquad \varphi(z, \cdot) = \Theta_q f(z, \cdot),$$

and

$$(12) \qquad f(z, \varsigma) = \frac{-1}{2\pi} \frac{1}{\varsigma - z} \prod_{j=1}^{2q-1} \frac{z - a_j}{\varsigma - a_j}.$$

Note that for $z \in \Lambda \backslash \{a_1, \ldots, a_{2q-1}\}$, $f(z, \cdot) \in R_q$. We let

$$\mathcal{F}_{1-q}(\Delta) = \{\text{restrictions to}\, \Lambda\, \text{of potentials}\, F_\psi \text{with}\, \psi \in \mathbf{A}_q(\Delta)\}.$$

As usual $\mathcal{F}_{1-q} = \mathcal{F}_{1-q}(\Omega)$. Observe that $\mathcal{F}_{1-q}(\Delta)$ is a finite-dimensional space of continuous functions on Λ. Also $\mathcal{F}_{1-q}(\Delta) \subset \mathcal{F}_{1-q}$, for all Δ.

If $f \in R_q$, then we can find $m \geq 1$ distinct points $b_1, \ldots, b_m$ in $\Lambda \backslash \{a_1, \ldots, a_{2q-1}\}$ and complex numbers $\beta_1, \ldots, \beta_m$ so that

$$(13) \qquad f(\varsigma) = \sum_{j=1}^{m} \beta_j f(b_j, \varsigma), \qquad \varsigma \in \mathbf{C}.$$

The points $b_1, \ldots, b_m$ and the constants $\beta_1, \ldots, \beta_m$ are uniquely determined by f. We now define a surjective linear map

$$\mathcal{K} : R_q \to \mathcal{F}_{1-q}^*$$

[3]If $\gamma_j(a_k) = \infty$, then holomorphicity at this point means $f(z) = O(|z|^{-2q})$, $z \to \infty$. Conventions regarding ∞ will henceforth be ignored.

from R_q to the dual space of $\mathcal{F}_{1-q}$ by the formula

$$\mathcal{K}(f)(F) = \sum_{j=1}^{m} \beta_j F(b_j), \qquad F \in \mathcal{F}_{1-q}, \tag{14}$$

where $f \in R_q$ is given by (13).

THEOREM 2. *Given $f \in R_q$, then*

$$\Theta_q f | \Delta = 0 \Leftrightarrow \mathcal{K}(f) | \mathcal{F}_{1-q}(\Delta) = 0. \tag{15}$$

The proof uses the duality given by the Petersson scalar product (4) and the identity (10).

Since $\mathcal{K}$ is a very simple operator, Theorem 2 shows that the vanishing problem is completely solved if we can construct a basis for $\mathcal{F}_{1-q}(\Delta)$.

3. Let $PH^1_\Delta(\Pi_{2q-2})$ denote the Eichler cohomology group of Δ-parabolic cohomology classes (see Kra [**5**, Chapter V]), where Π_{2q-2} is the space of polynomials of degree $\leq 2q-2$, and let $PH^1(\Pi_{2q-2})$ denote the space cohomology classes that are parabolic with respect to all parabolic elements of Γ. Given $\psi \in \mathbf{A}_q(\Delta)$, then

$$\gamma \mapsto F_\psi(\gamma)(\gamma')^{1-q} - F_\psi, \qquad \gamma \in \Gamma, \tag{16}$$

defines a cohomology class $\beta^*(\psi) \in PH^1(\Pi_{2q-2})$, known as the *Bers class* of ψ.

THEOREM 3. *If the Bers map*

$$\beta^* : \mathbf{A}_q \to PH^1(\Pi_{2q-2})$$

is surjective, then $\mathcal{F}_{1-q}$ can be determined algebraically from the parabolic Π_{2q-2}-cocycles for the group Γ.

We must explain what we mean by determining $\mathcal{F}_{1-q}$ algebraically. Let us assume that $a_1, \ldots, a_{2q-1}$ are fixed points of loxodromic elements of Γ. Theorem 3 means that we can construct algebraically the values at the loxodromic fixed points of functions $F_1, \ldots, F_d$ that form a basis for $\mathcal{F}_{1-q}$. In the proof, we use the fact that if the continuous function F on Λ represents the cocycle χ; that is, if

$$F(\gamma z)\gamma'(z)^{1-q} - F(z) = \chi(\gamma)(z), \qquad z \in \Lambda, \tag{17}$$

then for $b \in \Lambda$, a fixed point of a loxodromic element $g \in \Gamma$, we must have

$$F(b) = \chi(g)(b)[g'(b)^{1-q} - 1]^{-1}. \tag{18}$$

4. The map β^* of Theorem 3 is surjective for many geometrically finite function groups (Nakada [8]); in particular, for Fuchsian, quasi-Fuchsian, and Schottky groups. In principle, there is an algorithm for each such group to decide when $\Theta_q f = 0$ for a given $f \in R_q$. We state our most explicit construction of such an algorithm in

THEOREM 4. *Let Γ be a Schottky group or a finitely generated Fuchsian or quasi-Fuchsian group of the first kind given by a standard presentation on a canonical set of generators. Let $f \in R_q$ have poles only at loxodromic fixed points. Then we can write down a (finite) algorithm that determines whether or not $\Theta_q f = 0$.*

5. Let Γ be a finitely generated Fuchsian group of the first kind acting on the unit disk Δ. Then $\Lambda = \partial\Delta =$the unit circle, and $\Omega = \{z \in \mathbf{C} \mid |z| \neq 1\} \cup \{\infty\}$. To determine when a Poincaré series $\Theta_q f$, $f \in R_q$, vanishes identically only on Δ, we need to select $\mathcal{F}_{1-q}(\Delta)$ from $\mathcal{F}_{1-q}$. A not entirely satisfactory answer is contained in

THEOREM 5. *Let Γ be a finitely generated Fuchsian group of the first kind acting on the unit disk Δ. Then there exists an integer $n = n(q)$ such that for $F \in \mathcal{F}_{1-q}$, we have*

$$F \in \mathcal{F}_{1-q}(\Delta) \Leftrightarrow \int_0^{2\pi} e^{i(1-k-2q)\theta} F(e^{i\theta})\, d\theta = 0 \quad \textit{for } k = 0, 1, \ldots, n.$$

The debt of this paper to the fundamental contributions of Ahlfors [**1**] and Bers [**2**] is obvious, and I am delighted to acknowledge it. Hejhal's paper [**4**], which contains a somewhat less explicit solution to the vanishing problem for a more limited class of groups, was a useful reminder that this problem should have an algebraic solution. Our solution differs radically from Hejhal's. We rely in very basic ways on the Eichler cohomology machinery [**1**, **2**, **5**]. I am happy to thank M. Sheingorn for his insistence that the vanishing problem is important and interesting. Complete proofs and applications will appear elsewhere [**6**].

REFERENCES

1. L. V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964), 413–429; **87** (1965), 759.
2. L. Bers, *Inequalities for finitely generated Kleinian groups*, J. Analyse Math. **18** (1967), 23–41.
3. ——, *Poincaré series for Kleinian groups*, Comm. Pure Appl. Math. **26** (1973), 667–672; **27** (1974), 583.
4. D. A. Hejhal, *Monodromy groups and Poincaré series*, Bull. Amer. Math. Soc. **84** (1978), 339–376.
5. I. Kra, *Automorphic forms and Kleinian groups*, Benjamin, Reading, Mass., 1972.
6. ——, *On the vanishing of and spanning sets for Poincaré series for cusp forms* (to appear).
7. I. Kra and B. Maskit, *Bases for quadratic differentials*, Comment. Math. Helv. (to appear).
8. M. Nakada, *Quasi-conformal stability of finitely generated function groups*, Tôhoku Math. J. **30** (1978), 45–58.
9. H. Petersson, *Die linearen Relationen zwischen den ganzen Poincaréschen Reihen von reeller Dimension zur Modulgruppe*, Abh. Math. Sem. Hamb. Univ. **12** (1938), 415–472.
10. H. Poincaré, *Memoire sur les fonctions fuchsiennes*, Acta Math. **1** (1882), 193–294.
11. S. Wolpert, *The Fenchel-Nielsen deformation*, Ann. of Math. (2) **115** (1982), 501–528.

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