

SMOOTH EXTENDABILITY OF PROPER HOLOMORPHIC MAPPINGS

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In [9] Ch. Fefferman proved that any biholomorphic mapping $f: \Omega_1 \rightarrow \Omega_2$ between strictly pseudoconvex C^∞ -smooth domains in \mathbf{C}^n extends smoothly to the boundary. Subsequently, the proof of this result has been simplified considerably by S. Webster [14, 15], E. Ligočka [12], St. Bell [1, 2]. And it was St. Bell who realized the importance of the following regularity condition of the Bergman projection for the proof of such extendability results:

DEFINITION. A domain $\Omega \subset \subset \mathbf{C}^n$ is said to satisfy condition R for its Bergman projection operator P if for any positive integer s there is an integer N such that P is a bounded linear operator from $W_0^{s+N}(\Omega)$ to $H^s(\Omega)$.

(Here $W_0^s(\Omega)$ denotes as usual the closure of $C_0^\infty(\Omega)$ in the Sobolev s -Norm $\|\cdot\|_s$ with respect to the volume Lebesgue-measure on Ω and $H^s(\Omega)$ is the space of holomorphic functions on Ω with finite $\|\cdot\|_s$ -norm.)

Since condition R is a consequence of subelliptic estimates for the $\bar{\partial}$ -Neumann problem, it is known to be satisfied for instance in the following cases:

- (1) Ω strictly pseudoconvex, C^∞ -smooth (J. J. Kohn [10]);
- (2) Ω pseudoconvex, C^ω -smooth (J. J. Kohn [11], K. Diederich, J. E. Fornaess [6]).

The new methods allowed to generalize Fefferman's result. It is now known that a biholomorphic mapping $f: \Omega_1 \rightarrow \Omega_2$ extends smoothly up to the boundary if Ω_1 and Ω_2 are C^∞ -smooth and, in addition, both satisfy condition R [1] or both are pseudoconvex and at least one satisfies condition R [2].

The result which we wish to announce deals with the case of proper holomorphic mappings and is contained in the

THEOREM. *Let $\Omega_1, \Omega_2 \subset \subset \mathbf{C}^n$ be C^∞ -smooth pseudoconvex domains and suppose that Ω_1 satisfies condition R . Then any proper holomorphic mapping $f: \Omega_1 \rightarrow \Omega_2$ extends smoothly up to the boundary.*

For unbranched mappings f this result is contained in K. Diederich and J. E. Fornaess [7]. This also includes the case of Ω_1, Ω_2 being strictly pseudoconvex and f proper holomorphic since any such f is necessarily unbranched, S. Pincuk [13]. Under different, more restrictive assumptions on Ω_1 and Ω_2 the result

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was obtained by S. Bell in [4 and 5]. More detailed information about the history will be given in the paper containing also all the proofs.

Our proof is based on the following transformation formula for the Bergman projection proved by S. Bell [3]:

PROPOSITION 1. *Let $f: \Omega_1 \rightarrow \Omega_2$ be a proper holomorphic mapping between bounded domains in \mathbb{C}^n and let P_i denote the Bergman projection on Ω_i . Then one has*

$$P_1(u \cdot (h \circ f)) = u \cdot (P_2(h) \circ f)$$

for all $h \in L^2(\Omega_2)$. Here $u = \det f'$.

As a consequence of this, S. Bell [3] obtains

PROPOSITION 2. *Under the assumptions of the theorem one has $u \cdot h \circ f \in A^\infty(\Omega_1)$ for all $h \in A^\infty(\Omega_2)$, in particular $u \in A^\infty(\Omega_1)$.*

The difficulty in deriving the theorem from this statement is to show that the functions $u \cdot h \circ f \in A^\infty(\Omega_1)$ can be divided by u in $A^\infty(\Omega_1)$ even close to the cluster points of the branching locus X of f at $b\Omega_1$. This, certainly, would be hopeless if the Jacobian determinant u could vanish to infinite order at such points. But we can show

LEMMA 1. *In the situation of the theorem the Jacobian determinant u of f does not vanish to infinite order at any point of $\bar{\Omega}_1$.*

For the proof we assume that u does vanish to infinite order at $q \in b\Omega_1$ and take an arbitrary nontangential cone K in Ω_1 with vertex at q . The function

$$u_1(z) := \prod_{\zeta \in f^{-1} \circ f(z)} u(\zeta) \quad \text{for } \zeta \in \Omega_1 \setminus X$$

extends to a bounded holomorphic function on Ω_1 which vanishes in Ω_1 exactly on $X_1 := f^{-1}f(X)$ and at q goes to zero with infinite order. There is a bounded holomorphic function u_2 on Ω_2 with $u_1 = u_2 \circ f$ and $\{u_2 = 0\} = f(X_1) = X_2$. Notice, that on $\Omega_2 \setminus X_2$ there are locally defined inverse mappings F_1, \dots, F_m of f , the Jacobian determinants of which are denoted by U_k . One has

$$(1) \quad u_2 \cdot \prod_{k=1}^m U_k \equiv 1.$$

Furthermore, it is well known that there are an $\eta > 0$ and positive constants c, C such that

$$(2) \quad c \text{dist}^{1/\eta}(z; b\Omega_1) \leq \text{dist}(f(z), b\Omega_2) \leq C \text{dist}^\eta(z, b\Omega_1)$$

on Ω_1 . By considering for each fixed $k = 1, \dots, m$, the elementary symmetric functions of the k th coordinate functions of $F_j, j = 1, \dots, m$, which are globally

defined and bounded on Ω_2 , applying the Schwarz lemma to them and studying the zeros of the polynomial in one variable with these coefficients we can show using (2) that for large positive integers N the preimage of a polynomial neighborhood of X_2 of the form

$$V_N^2 := \{w \in \Omega_2 : \text{dist}(w, X_2) < \text{dist}^N(w, b\Omega_2)\}$$

is contained in a polynomial neighborhood V_M^1 of X_1 in Ω_1 of the same form. Here $M < N$, but M goes to infinity if N does. Next by using the Blaschke condition for the zero set X_1 of u_1 one proves that for large M

$$(3) \quad \lim_{r \rightarrow 0} \frac{A(V_M^1 \cap K \cap B_r)}{A(K \cap B_r)} = 0$$

where $A(\)$ means euclidean volume and B_r is the ball of radius r around q . Therefore, there is a sequence $(z^k) \subset K, z^k \rightarrow q$, such that $\Omega_2 \setminus X_2$ contains for each k , a relatively “large” ball around $f(z^k)$. This allows us to show that all $U_j(f(z^k))$ have to be small compared to a fixed negative power of

$$\text{dist}(f(z^k), b\Omega_2).$$

Because of (1) and (2) we obtain that $u_1(z^k)$ stays above a certain fixed power of $\text{dist}(z^k, b\Omega_1)$. This contradicts our assumption. \square

The next step is the division by u . We show

LEMMA 2. *Let h be a bounded holomorphic function on Ω_1 such that $u \cdot h^N \in A^\infty(\Omega_1)$ for all positive integers N , then $h \in A^\infty(\Omega_1)$.*

This is proved by a rather technical inductive procedure. We will, therefore indicate here only how the continuity of h follows. We fix a point $q \in b\Omega_1$ and a generic transverse complex line H through q such that $u \mid H \cap \Omega_1$ vanishes at q only to finite order k . We can assume that $q = 0$ and

$$H = \{z_2 = \dots = z_n = 0\}.$$

For $p \in b\Omega_1$ near 0 we denote by $H(p)$ the complex line through p parallel to H . It is easy to see that $h \mid \Omega_1 \cap H_p$ is C^∞ up to $b\Omega_1 \cap H_p$ near 0. This gives us an extension of h to $b\Omega_1$ near 0. We may assume that $h(0) = 0$. We denote

$$g^{(s)} := \partial^s g / \partial z_1^s$$

and use the following observation.

LEMMA 3. *For each k there exists a sequence of polynomials $P_j, j = 1, 2, 3, \dots$, in u and h and their derivatives, such that for all $N > k$*

$$(4) \quad (uh^N)^{(k)} = u^{(k)}h^N + \left(\sum_{j=1}^k N^j P_j \right) h^{N-k}.$$

The proof goes by induction over k and is straightforward. Let us now assume that there is a sequence $S = (p_s) \subset \Omega_1$, $p_s \rightarrow 0$, such that $h(p_s) \rightarrow \hat{\gamma} \neq 0$. For $N > k$ we have

$$(5) \quad (uh^N)^{(k)} \rightarrow 0 \quad \text{on } S,$$

since this function is in A^∞ and its restriction to $H(0) \cap \Omega_1$ goes to zero at 0. Therefore, we get from (4)

$$(6) \quad u^{(k)}h^k + \left(\sum_{j=1}^k N^j P_j \right) \rightarrow 0 \quad \text{on } S.$$

Next we prove inductively for $0 \leq \kappa \leq k$:

(A_κ) There are coefficients $a_{j,s}^{(\kappa)}$, $j = 1, \dots, \kappa$, $s = 1, 2, 3, \dots$, such that for all $N > k$

$$u^{(\kappa)}h^\kappa(p_s) + \sum_{j=1}^{\kappa} N^j a_{j,s}^{(\kappa)} \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

For $\kappa = k$ the choice $a_{j,s}^{(k)} := P_j(p_s)$ satisfies (A_k) because of (6). If (A_κ) for some $1 \leq \kappa \leq k$ has been reached, one may put

$$a_{j,s}^{(\kappa-1)} := a_{j,s}^{(\kappa)}(1 - 2^j 2^{-\kappa})(1 - 2^{-\kappa})^{-1}.$$

That this satisfies ($A_{\kappa-1}$) follows from the fact that

$$(1 - 2^{-\kappa})^{-1} \left[u^{(\kappa)}h^\kappa(p_s) + \sum_{j=1}^{\kappa} N^j a_{j,s}^{(\kappa)} - 2^{-\kappa} \left(u^{(\kappa)}h^\kappa(p_s) + \sum_{j=1}^{\kappa} (2N)^j a_{j,s}^{(\kappa)} \right) \right] \rightarrow 0$$

as $s \rightarrow \infty$. The statement (A_0) now gives $u^{(k)}h^k \rightarrow 0$ on S , a contradiction to $h \rightarrow \hat{\gamma} \neq 0$ on S . \square

An easy consequence of our theorem is the following statement. An independent and simpler proof of it has been given in [8].

COROLLARY. *Let $f: \Omega_1 \rightarrow \Omega_2$ be proper holomorphic, Ω_1, Ω_2 C^∞ -smooth and pseudoconvex. Suppose that Ω_1 satisfies condition R (e.g. Ω_1 strictly pseudoconvex). Then the branching locus of f does not cluster at any strictly pseudoconvex boundary point. In particular, f is unbranched if the $(2n - 3)$ -dimensional Hausdorff measure of the set of weakly pseudoconvex boundary points of Ω_1 is zero.*

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