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BOOK REVIEWS

Families of meromorphic functions on compact Riemann surfaces, by Makoto Namba, Lecture Notes in Math., vol. 767, Springer-Verlag, Berlin and New York, 1979, xii + 284 pp., $16.30.

The theory of compact Riemann surfaces, or equivalently of nonsingular algebraic curves over the complex numbers, has long been a rich and rewarding field of study and remains a surprisingly lively area of current research interest. Among the problems still actively being investigated are a number involving divisors and their associated meromorphic functions, following the trail blazed by Riemann, Abel, Jacobi, and many others. Recall that a divisor on an algebraic curve M is an element of the free abelian group generated by the points of M, or in other words is a finite sum $D = \Sigma_j n_j P_j$ where $n_j \in \mathbf{Z}$ and $P_j \in M$; such a divisor is called positive or effective and written $D \geqslant 0$ if each $n_j \geqslant 0$, and the degree of the divisor is the integer $\deg D = \Sigma_j n_j$. To any meromorphic function f not identically zero on M there is associated its divisor $D(f) = \Sigma_j n_j P_j$, where n_j is the order of f at the point P_j; $n_j > 0$ if f has a zero of order n_j at P_j, $n_j < 0$ if f has a pole of order $|n_j|$ at P_j, and points at which f is of order 0 are usually not listed. If $D(f) = \Sigma_j n_j P_j$ then $\deg D = \Sigma_j n_j = 0$ and $\frac{1}{2}\Sigma_j |n_j|$ is the order of the function f; a function f of order n when viewed as a holomorphic mapping $f: M \to \mathbf{P}^1$ exhibits M as an n-sheeted branched covering of the Riemann sphere $\mathbf{P}^1$. Conversely it is traditional to associate to any divisor D the complex vector space $L(D)$ consisting of the zero function together with all those meromorphic functions f on M such that $D(f) + D \geqslant 0$; thus if $f \neq 0$ then $f \in L(\Sigma_j n_j P_j)$ if the singularities of f are at most poles of order n_j at those points P_j for which $n_j > 0$ and f has zeros of order at least $|n_j|$ at those points P_j for which $n_j < 0$. The dimension of the projective space associated to $L(D)$ is called the dimension of the divisor D and is denoted by $\dim D$, so that $\dim D = \dim_{\mathbf{C}} L(D) - 1$; considering the associated projective space rather than $L(D)$ itself really amounts to emphasizing the divisors of the functions rather than the functions themselves, since $D(f) = D(cf)$ whenever $c \in \mathbf{C}$ and $c \neq 0$. If D is a positive divisor with $\deg D = n \geqslant 2g - 1$ where g is the genus of the curve M then it follows from the Riemann-Roch theorem that $\dim D = n - g$; however if $0 < n < 2g - 1$ then $\dim D$ is a subtle and quite nontrivial function of the divisor D and the curve M.

For instance if D is a positive divisor on M with $\deg D = n$ and $\dim D \geqslant 1$ then there are at least 2 linearly independent meromorphic functions in $L(D)$, so one of them must be a nonconstant function of some order $k \leqslant n$; thus if there exists on M a divisor $D \geqslant 0$ with $\deg D = n$ and $\dim D \geqslant 1$ then M can be represented as a k-sheeted branched covering of $\mathbf{P}^1$ for some $k \leqslant n$. If M has genus $g > 1$ then $\dim D \geqslant 1$ for any divisor $D \geqslant 0$ with $\deg D \geqslant 2g - 1$; it had long been asserted in the literature that on any curve of genus g there exists a divisor $D \geqslant 0$ with $\deg D \leqslant (g + 3)/2$ and $\dim D \geqslant 1$, but the first complete proof was only given in 1960 by T. Meis, **[20]**. It follows rather easily from the Riemann-Roch theorem that on any curve of genus g there are actually meromorphic functions of every order $k \geqslant g + 1$, but it is a nontrivial and indeed not fully solved problem to determine just which integers $k \leqslant g$ can be the orders of meromorphic functions on a given curve; some results in this direction are discussed in the book being reviewed here, and more can be found in a recent paper by G. Martens, **[16]**.

For the more detailed study of divisors and their associated meromorphic functions on a curve M it is convenient to introduce the Jacobi variety of M. If M has genus $g \geqslant 1$ then to any choice of bases $\{\omega_1, \ldots, \omega_g\}$ for the space of holomorphic differential 1-forms on M and $\{\lambda_1, \ldots, \lambda_{2g}\}$ for the homology group $H_1(M, \mathbf{Z})$ there is a corresponding period matrix $\Omega = \{\omega_{ij}\}$, the $g \times 2g$ matrix with entries $\omega_{ij} = \int_{\lambda_j} \omega_i$; the columns of this matrix generate a lattice subgroup $\mathfrak{L} = \Omega\mathbf{Z}^{2g} \subset \mathbf{C}^g$, and the quotient group $\mathbf{C}^g/\mathfrak{L} = J(M)$ is a compact complex algebraic torus, the Jacobi variety of M. Fixing a base point $P_0 \in M$, a holomorphic mapping $\phi\colon M \to J(M)$ can be defined by assigning to any point $P \in M$ the point $\phi(P) \in J(M)$ represented by the vector $\{\phi_i(P)\} \in \mathbf{C}^g$ with components $\phi_i(P) = \int_\lambda \omega_i$ where λ is some path from P_0 to P; changing the path λ changes the vector $\{\phi_i(P)\}$ only by an element of $\mathfrak{L}$, so the mapping ϕ is well defined. This mapping can then be extended to a homomorphism from the group of divisors on M to $J(M)$ by setting $\phi(\Sigma_j n_j P_j) = \Sigma_j n_j \phi(P_j)$. The basic property of the resulting homomorphism is Abel's theorem $\phi(D) = \phi(D')$, $\deg D = \deg D'$, precisely when the divisors D and D' are linearly equivalent, meaning that $D - D' = D(f)$ for some meromorphic function f on M. Clearly linearly equivalent divisors have the same degree and dimension; the homomorphism ϕ thus establishes a one-to-one correspondence between the set of linear equivalence classes of positive divisors of degree n and dimension r on M and a certain subset $G_n^r \subseteq J(M)$.

The subsets G_n^r are holomorphic subvarieties of $J(M)$ called the subvarieties of special positive divisors, and a good deal of effort has been expended in the study of them. They are really only of interest for the range of values $0 < n < 2g - 2$, so that restriction will be assumed henceforth without further mention. The first question is naturally whether these subsets are empty; for the special case $r = 1$ that is just the question whether there exists on M a divisor $D \geqslant 0$ of degree n with $\dim D \geqslant 1$, and has already been discussed. A consequence of the Riemann-Roch theorem known as Clifford's theorem asserts first that $G_n^r = \varnothing$ whenever $2r > n$, and second that $G_{2r}^r \neq \varnothing$ if and only if the curve M is hyperelliptic, that is, can be represented as a two-sheeted covering of $\mathbf{P}^1$. The conditions that $G_n^r \neq \varnothing$ for successively lower values of r put successively weaker restrictions on the curve M, although the precise restrictions seem not yet to have been sorted out completely. Finally there is the traditional assertion in the literature that, on any curve of genus g, whenever $(r + 1)(n - r) - rg \geqslant 0$ then $G_n^r \neq \varnothing$; complete proofs of that assertion for general values of r were finally given in the early 1970's by G. Kempf **[9]** and by S. Kleiman and D. Laksov **[11]**. The next question is what are the dimensions of these subvarieties, a question inspired by Riemann's assertion that the set of special linear series of degree n and dimension r on a curve of genus g depends on $(r + 1)(n - r) - rg$ parameters. That each irreducible component of G_n^r has dimension $\geqslant (r + 1)(n - r) - rg$ if $G_n^r \neq \varnothing$ was demonstrated by A. Brill and M. Noether in 1874 [**2**], while H. H. Martens showed in 1967 [**17**] that dim $G_n^r \leqslant n - 2r$ and gave an alternative proof of the Brill-Noether result, and related results were obtained by A. Mayer and R. S. Hamilton [**7**]. The conditions that the subvarieties G_n^r have components of dimensions d for successively lower values of d in the range $(r + 1)(n - r) - rg \leqslant d \leqslant n - 2r$ impose what are in some sense successively weaker restrictions on the curve M; H. H. Martens also showed in [**17**] that G_n^r has a component of the maximal dimension $n - 2r$ if and only if M is hyperelliptic and further results have subsequently been obtained by D. Mumford [**21**] and G. Martens [**16**], but the complete picture is far from clear. Finally the classical assertion in the literature is that on a generic curve of genus g the subvariety G_n^r is of pure dimension $(r + 1)(n - r) - rg$; special instances of that assertion were proved by H. Farkas [**4**], R. Lax [**13**], and S. Kleiman and D. Laksov [**12**], while the general result has recently been demonstrated in joint work by P. Griffiths, E. Arbarello, M. Cornalba, and J. Harris [**5**]. The last question to be mentioned here is what are the singularities of these subvarieties G_n^r; that question is inspired by the Riemann vanishing theorem, the theorem that for a surface of genus g the subvariety G_{g-1}^r consists of the points of multiplicity $r + 1$ on G_{g-1}^0. Several satisfactory proofs of this result are now available, [**15**], [**18**], [**19**]. It was demonstrated by A. Weil [**25**] that G_n^1 is precisely the singular locus of G_n^0 for $1 \leqslant n \leqslant g - 1$, and it is easy to see that G_n^{r+1} is contained in the singular locus of G_n^r whenever the latter is a proper subvariety of $J(M)$, [**6**]; however H. H. Martens showed that for special curves there are singularities of G_n^1 outside G_n^2 [**17**]. The most detailed results now known about the singularities of the subvarieties G_n^0 for $1 \leqslant n \leqslant g - 1$, extending the Riemann vanishing theorem to these subvarieties and describing the tangent cones at the singularities, are due to G. Kempf, [**9**], [**10**]. The singularities of the subvarieties G_n^r for $r > 0$ remain much less well understood.

The development that the theory of Riemann surfaces has undergone suggests another approach to the study of the subvarieties G_n^r, an approach which has so far been surprisingly little pursued but which is that of the book being reviewed here. Riemann asserted that curves of genus $g > 1$ depend on $3g - 3$ parameters; making this assertion precise has occupied many mathematicians ever since, and has inspired an extensive theory of deformations and families of complex and other structures. What is by now well established is that the set of nonsingular curves of genus $g > 1$ can be put into one-to-one correspondence with the points of a $(3g - 3)$-dimensional complex analytic variety M_g, the space of moduli of curves of genus g; moreover there is a $(3g - 2)$-dimensional complex analytic variety C_g, the universal curve of genus g, with a holomorphic mapping $\pi\colon C_g \to M_g$ such that $\pi^{-1}(t)$ is the curve represented by the point t for every $t \in M_g$. From an analytic point of view it is often more convenient to consider not just the curves themselves but the curves together with markings, which are either (i) canonical generators of their fundamental groups or (ii) bases for their first homology groups. In both cases the set of marked nonsingular curves of genus $g > 1$ can be put into one-to-one correspondence with the points of a $(3g - 3)$-dimensional complex analytic variety, the Teichmüller space T_g in case (i) and the Torelli space T_g' in case (ii); moreover in both cases there are $(3g - 2)$-dimensional varieties analogous to the universal curve of genus g, the universal curves over Teichmüller and Torelli spaces respectively. One advantage in considering Teichmüller space is that T_g itself and the universal curve over T_g are both actually complex manifolds, and the mapping between them is a nonsingular holomorphic mapping; Teichmüller space is a branched analytic covering space over Torelli space and over the moduli space. A more detailed discussion and historical survey can be found in [22] or [24].

In addition to the universal curve over Teichmüller space there is a $(4g - 3)$-dimensional complex manifold J_g, the universal Jacobi variety, with a nonsingular holomorphic mapping $\pi: J_g \to T_g$ such that $\pi^{-1}(t)$ is the Jacobi variety of the curve represented by t for every point $t \in M_g$, and the universal curve can be imbedded in J_g compatibly with the imbedding of any marked curve in its Jacobi variety, [3]. The union of the subvarieties $G_n^r(t) \subset \pi^{-1}(t)$ as t varies over T_g is an analytic subvariety $G_n^r \subseteq J_g$, the universal space of special positive subvarieties, about which many questions can be asked but few have been answered: what is the dimension of G_n^r, how many irreducible components does it have, what are its singularities, how are the singularities of G_n^r related to the singularities of the fibres $G_n^r(t)$, and so on? Particular cases of these questions involving properties of Weierstrass points on the universal curve have been investigated in this context, [23], [1], [14], [8]. Much remains to be done though.

The book under review here is an essay in the application of the recently developed theories of deformations and families of complex analytic manifolds to the study of divisors and their associated meromorphic functions on Riemann surfaces, discussing the currently available machinery most likely to be applicable to the problems just mentioned and illustrating its possible utility with a variety of interesting results; for instance, $G_n^1 \subseteq J_g$ is a nonsingular subvariety of dimension $2n + 2g - 5$ for $1 \leqslant n \leqslant g$. The principal application of the machinery discussed here though is to an investigation of the set $R_n(M)$ of all meromorphic functions of order equal to n on a curve M of genus g, and of the union $R_n = \bigcup_{M \in T_g} R_n(M)$ of these sets over the Teichmüller space T_g. The sets $R_n(M)$ are shown to be analytic varieties, some of which actually have singularities, while the sets R_n for $n \geqslant 2$ are nonsingular complex varieties of dimensions $2n + 2g - 2$. The book presupposes some familiarity with Riemann surfaces, while the theory of deformation of complex manifolds is quickly reviewed so is not really a prerequisite; some acquaintance with the latter theory is helpful though, since it is easy to get lost in the technicalities necessarily involved. There are many fascinating open problems in this area, and I heartily second Professor Namba's hope that his work will incite interest in solving them.

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ROBERT C. GUNNING

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Operator colligations in Hilbert spaces, by Mikhail S. Livshits [Moshe Livsic] and Artem A. Yantsevich, Winston, Washington, D. C. (distributed by Wiley, New York), 1979, xii + 212 pp., $19.95.

The general philosophy behind the idea of *operator models* as a tool for studying a bounded linear operator on a Hilbert space is to associate with