BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 2, March 1981

SINGULAR INTEGRALS ON PRODUCT DOMAINS

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Introduction. In their well-known theory of singular integrals on R^n, Calderón and Zygmund [1] obtained the boundedness of certain convolution operators on R^n which generalize the Hilbert transform on R^1. Thus, we know that if $Tf = f * K$ and $K(x)$ is defined on R^n and satisfies the analogous estimates that $1/x$ satisfies on R^1, namely

(i) $|K(x)| \leqslant C/|x|^n$,

(ii) $\int_{\alpha<|x|<\beta} K(x)dx = 0$ for all $0 < \alpha < \beta$,

(iii) $\int_{|x|>2|h|} |K(x + h) - K(x)|dx \leqslant C$ for all $h \neq 0$,

then T is a bounded operator on $L^p(R^n)$ for $1 < p < \infty$. (See Stein [2].)

Now if we take the space $R^n \times R^m$ along with the *two* parameter family of dilations $(x, y) \longrightarrow (\delta_1 x, \delta_2 y)$, $x \in R^n$, $y \in R^m$, $\delta_i > 0$, instead of the usual one parameter dilations, we are led to consider operators which generalize the double Hilbert transform on R^n, $Hf = f * 1/xy$. The boundedness properties of H are usually very easy to obtain by an argument which iterates the one-dimensional theory of the Hilbert transform. But if we consider, more generally, operators $Tf = f * K$ where K satisfies analogous estimates to those satisfied by $1/xy$ but cannot be written in the form $K_1(x) \cdot K_2(y)$ then the argument which deals with H fails.

We wish to announce here that for various classes of kernels K which "look like" $1/xy$ on R^2, but are not products of two functions on the x and y variables respectively, the convolution operators are bounded on L^p for $1 < p < \infty$ and take $L \log^+ L(R^n \times R^m)$ boundedly to weak L^1. In particular this involves the problem of formulating the right two parameter versions of the assumptions on the kernel K.

We wish to take this opportunity to thank E. M. Stein for his help in the course of this work. The formulation of several of our theorems follows his suggestions, and Theorem 3 in its L^p form is due to him. We also wish to thank the Institute for Advanced Study for its hospitality and the National Science Foundation for its financial support.

Statement of results. We shall state three results dealing with the action of convolution operators. These deal with the action of these operators on L^2, L^p for $1 < p < \infty$, and $L \log^+ L$ respectively.

Received by the editors June 18, 1980.
1980 *Mathematics Subject Classification*. Primary 42A40.

Before stating these results let us agree on the following notation: if $K(x, y)$ is a kernel on $R^n \times R^m$ then set

$$\Delta_h^1(K)(x, y) = K(x + h, y) - K(x, y),$$

$$\Delta_k^2(K)(x, y) = K(x, y + k) - K(x, y),$$

$$\Delta_{h,k}^{1,2}(K)(x, y) = \Delta_k^2(\Delta_h^1(K))(x, y).$$

THEOREM 1. *Let $K(x, y)$ be defined on $R^n \times R^m$ and satisfy*

(1) $|K(x, y)| \leqslant C/|x|^n|y|^m$,

(2) $$\iint_{\substack{|x|>2|h| \\ |y|>2|k|}} |\Delta_{h,k}^{1,2}(K)(x, y)|\, dxdy \leqslant C,$$

$$\int_{|x|>2|h|} |\Delta_h^1(K)(x, y)|\, dx \leqslant \frac{C}{|y|^m}, \qquad \int_{|y|>2|k|} |\Delta_k^2(K)(x, y)|\, dy \leqslant \frac{C}{|x|^n},$$

(3) $$\left| \iint_{\substack{\alpha_1<|x|<\alpha_2 \\ \beta_1<|y|<\beta_2}} K(x, y)\, dxdy \right| \leqslant C \quad \text{for all } 0 < \alpha_1 < \alpha_2,\ 0 < \beta_1 < \beta_2,$$

$$\left| \int_{\alpha_1<|x|<\alpha_2} K(x, y)\, dx \right| \leqslant \frac{C}{|y|^m} \quad \text{for all } 0 < \alpha_1 < \alpha_2,$$

$$\left| \int_{\beta_1<|y|<\beta_2} K(x, y)\, dy \right| \leqslant \frac{C}{|x|^n} \quad \text{for all } 0 < \beta_1 < \beta_2$$

and

(4) *if* $G(x) = \int_{\beta_1<|y|<\beta_2} K(x, y)dy$, *then*

$$\int_{|x|>2|h|} |G(x + h) - G(x)|dx \leqslant C$$

uniformly in β_1 and β_2 and if

$$H(y) = \int_{\alpha_1<|x|<\alpha_2} K(x, y)\, dx$$

then

$$\int_{|y|>2|k|} |H(y + k) - H(y)|\, dy \leqslant C$$

uniformly in α_1 and α_2. Then

$$\|f * K\|_{L^2(R^n \times R^m)} \leqslant C'\|f\|_{L^2(R^n \times R^m)}.$$

Let us introduce the terminology that a class, $\mathcal{C}$, of kernels on $R^n \times R^m$ will be called "invariant under dilations" provided that whenever $K(x, y) \in \mathcal{C}$ then $\delta_1^{-n}\delta_2^{-m}K(x/\delta_1, y/\delta_2) \in \mathcal{C}$.

THEOREM 2. *Suppose* $\mathcal{C}$ *is a class of kernels invariant under dilations of* $R^n \times R^m$. *Assume that any* $K \in \mathcal{C}$ *satisfies*

(1) $$\iint_{|x|,|y|\leq 1} |K(x, y)|\, |x|\, |y|\, dxdy \leq C$$

where C *is independent of* $K \in \mathcal{C}$,

(2) $$\int_{\alpha_1<|x|<\alpha_2} K(x, y)dx = 0, \qquad y \in R^m,$$

$$\int_{\beta_1<|y|<\beta_2} K(x, y)dy = 0, \qquad x \in R^n,$$

and

(3) *for some* $\alpha > 0$,

$$\iint_{|x|,|y|>1} |\Delta^{1,2}_{h,k}(K)(x, y)|^2 dxdy \leq C|h|^\alpha |k|^\alpha$$

for $|h|$, $|k| < ½$ *with* C *independent of* $K \in \mathcal{C}$,

$$\int_{|x|>1} \left(\int_{|y|<1} |\Delta^1_h(K)(x, y)|\, |y|\, dy\right)^2 dx \leq C|h|^\alpha,$$

$$\int_{|y|>1} \left(\int_{|x|<1} |\Delta^2_k(K)(x, y)|\, |x|\, dx\right)^2 dy \leq C|k|^\alpha.$$

Then $\|f * K\|_{L^p(R^n\times R^m)} \leq C_p'\|f\|_{L^p(R^n\times R^m)}$, $1 < p < \infty$, *whenever* $K \in \mathcal{C}$ *and where* C_p' *depends on* p, n *and* C.

THEOREM 3. *Suppose* $K(x, y)$ *is defined on* R^2 *and is* C^∞ *away from the coordinate axes, and satisfies*

(1) $$\left|\frac{\partial^{\alpha+\beta}K(x, y)}{\partial x^\alpha \partial y^\beta}\right| \leq \frac{C}{|x|^{\alpha+1}|y|^{\beta+1}}$$

for all $\alpha, \beta \geq 0$,

(2) $$\left|\iint_{\substack{\sigma_1<|x|<\sigma_2\\ \tau_1<|y|<\tau_2}} K(x, y)dxdy\right| \leq C,$$

$$\left|\int_{\sigma_1<|x|<\sigma_2} K(x, y)dx\right| \leq \frac{C}{|y|} \quad \left|\int_{\tau_1<|y|<\tau_2} K(x, y)dy\right| \leq \frac{C}{|x|}$$

and

(3) *if* $G(x) = \int_{\tau_1<|y|<\tau_2} K(x, y)dy$, $H(y) = \int_{\sigma_1<|x|<\sigma_2} K(x, y)dx$ *then* $G, H \in C^\infty$ *and*

$$\left|\frac{\partial^\gamma G}{\partial x^\gamma}\right| \leq \frac{C}{|x|^{\gamma+1}}, \quad \gamma \geq 0, \qquad \left|\frac{\partial^\gamma H}{\partial y^\gamma}\right| \leq \frac{C}{|y|^{\gamma+1}}$$

where C *is independent of* σ_1, σ_2, τ_1 *and* τ_2. *Then if* $Tf = f * K$, T *maps* $L(\log^+ L)(R^2)$ *boundedly into weak* L^1.

We should remark at this point that the hypotheses of Theorems 1 and 2 are valid in case

$$K(y, x) = \frac{\Omega(x, y)}{|x|^n |y|^m},$$

$$\Omega(\delta_1 x, \delta_2 y) = \Omega(x, y)\ \delta_i > 0, \qquad \int_{S^{n-1}} \Omega(x, y)\, d\sigma(x) = 0$$

and if $\Omega_x(y) = \Omega(x, y)$ then the map $x \longrightarrow \Omega_x$ from S^{n-1} to the Lipschitz class $\Lambda^\alpha(S^{m-1})$ is Λ^α, with the symmetric conditions exchanging x and y.

2. $K(x, y)$ is C^2 away from $\{(x, y) | x = 0 \text{ or } y = 0\}$ and

$$(\alpha)\ |K(x, y)| \leqslant \frac{C}{|x|^n |y|^m},$$

$$(\beta)\ |\nabla_x K(x, y)| \leqslant \frac{C}{|x|^{n+1} |y|^m}\quad |\nabla_y K(x, y)| \leqslant \frac{C}{|x|^n |y|^{m+1}},$$

$$|\nabla_x \nabla_y K(x, y)| \leqslant \frac{C}{|x|^{n+1} |y|^{m+1}}$$

and

$$(\gamma)\ \int_{\alpha<|y|<\beta} K(x, y)\,dy = \int_{\alpha<|x|<\beta} K(x, y)\,dx = 0, \qquad \forall\, 0 < \alpha < \beta.$$

Sketch of ideas for proofs of theorems. For the L^2 case, we show that under the assumptions of Theorem 1, the kernel K has bounded Fourier transform. By dilation invariance of the properties of K we may estimate $K(\xi, \eta)$ assuming $|\xi| = |\eta| = 1$. To do this we write

$$\hat{K}(\xi, \eta) = \iint K(x, y) e^{i(x\cdot\xi + y\cdot\eta)} dx\,dy$$

$$= \iint_{|x|,|y|>10} K_1 + \iint_{\substack{|x|\leqslant 10\\ |y|<10}} K_1 + \iint_{\substack{|x|>10\\ |y|\leqslant 10}} K_1 + \iint_{|x|,|y|\leqslant 10} K_1$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

and $K_1(x, y) = K(x, y) e^{i(\xi\cdot x + \eta\cdot y)}$. We show how to estimate III to illustrate the method for all the terms.

$$\mathrm{III} = -\iint_{\substack{|x|>10\\ |y|\leqslant 10}} K(x, y)(1 - e^{iy\cdot\eta}) e^{ix\cdot\xi} dx\,dy + \iint_{\substack{|x|>10\\ |y|\leqslant 10}} K(x, y) e^{ix\cdot\xi} dx\,dy$$

$$= \mathrm{III}_{\mathrm{I}} + \mathrm{III}_{\mathrm{II}}.$$

To estimate $\mathrm{III}_{\mathrm{I}}$, notice for each fixed y, the x integral equals

$$\int_{|x|>10} [K(x + \pi\xi, y) - K(x, y)]\, e^{i\xi\cdot x} dx$$

which is $\leqslant C/|y|^m$ + an error where the error takes the form

$$\int_{x\in S} K(x, y)e^{i\xi\cdot x}dx, \qquad S \subset \{x|\, 6 < |x| < 14\}.$$

So the error $\leqslant C/|y|^m$ (since $|K(x, y)| \leqslant C/|x|^n|y|^m$). Then

$$\mathrm{III}_{\mathrm{I}} \leqslant \int_{|y|\leqslant 10} \frac{C}{|y|^m}\, |1 - e^{iy\cdot\eta}|dy \leqslant C'.$$

To handle $\mathrm{III}_{\mathrm{II}}$ we write in the notation of Theorem 1,

$$\mathrm{III}_{\mathrm{II}} = \int_{|x|>10} G(x)e^{i\xi\cdot x}dx$$

$$\leqslant \int_{|x|>10} |G(x + \pi\xi) - G(x)|dx + 2\int_S |G(x)|\, dx$$

where $S \subset \{x\,|\, 6 \leqslant |x| < 14\}$,

$$\leqslant C + 2\int_S \frac{C}{|x|^n} dx \leqslant C'.$$

This concludes the estimate of term III in the expression for $\hat{K}$.

Let us move now to sketching the idea of the proof for Theorem 2 dealing with the L^p theory.

The idea of the proof of Theorem 2 is to prove an inequality like $S(Tf)\,(x, y) \leqslant C' g_\lambda^*(f)\,(x, y)$ where S and g_λ^* are two parameter versions of the familiar Littlewood-Paley functions (see Stein [2]). Because of dilation invariance arguments in order to prove such an inequality we need only show that

$$\mathrm{I} = \iint_{R^n\times R^m} |K * \varphi(x, y)|^2(1 + |x|)^{n+\epsilon}(1 + |y|)^{m+\epsilon}dxdy < \infty$$

for some suitably chosen nontrivial function φ. (See Stein [2] or Calderón-Torchinsky [3] where such facts are used to prove Hörmander-Marcinkiewicz type multiplier theorems in the 1-parameter case.) To estimate I, write

$$\mathrm{I} = \iint_{|x|,|y|<10} H + \iint_{\substack{|x|\leqslant 10\\ |y|>10}} H + \iint_{\substack{|x|>10\\ |y|\leqslant 10}} H + \iint_{|x|,|y|>10}$$

$$= (1) + (2) + (3) + (4)$$

where $H(x, y) = |K * \varphi(x, y)|^2(1 + |x|)^{n+\epsilon}(1 + |y|)^{m+\epsilon}$. Let us show how to handle (4). For $|x|, |y| > 10$,

$$K * \varphi(x, y) = \iint_{R^n\times R^m} K(x - h, y - k)\varphi(h, k)\, dhdk,$$

and we shall choose $\varphi \in C^\infty$ so that support $\varphi \subset \{(x, y)|\, |x| < 1, |y| < 1\}$ and $\int_{R^n}\varphi(x, y)dx = 0\ \forall\, y \in R^m$ $\int_{R^m}\varphi(x, y)dx = 0\ \forall x \in R^n$. Then using these properties of φ we see that

$$K * \varphi(x, y) = \iint_{|h|,|k|<1} \Delta_{h,k}^{1,2}(K)(x, y)\varphi(h, k)\, dhdk,$$

and we see that

$$\iint_{|x|,|y|>10} |K * \varphi|^2 (1 + |x|)^{n+\epsilon}(1 + |y|)^{m+\epsilon} dxdy$$

$$\leqslant \iint_{|x|,|y|>10} \left(\iint_{|h|,|k|\leqslant 1} |\Delta^{1,2}_{h,k}(K)(x, y)|\, dhdk\right)^2 \cdot (1 + |x|)^{n+\epsilon}(1 + |y|)^{m+\epsilon} dxdy$$

$$\leqslant \iint_{|h|,|k|\leqslant 1} \left(\iint_{|x|,|y|>10} |\Delta^{1,2}_{h,k}(K)(x, y)|^2 \cdot (1 + |x|)^{n+\epsilon}(1 + |y|)^{m+\epsilon} dxdy\right) dhdk.$$

To show that this is finite we shall estimate the inner integral independent of h and k:

$$\iint_{|x|,|y|>10} |\Delta^{1,2}_{h,k}(K)|^2 (x, y)|x|^{n+\epsilon}|y|^{m+\epsilon} dxdy$$

$$\leqslant (*) = C \sum_{l\geqslant 0} \sum_{j\geqslant 0} \iint_{\substack{|x|\sim 2^j \\ |y|\sim 2^l}} |\Delta^{1,2}_{h,k}(K)|^2 (x, y)\, dxdy \cdot 2^{j(n+\epsilon)} 2^{l(m+\epsilon)}$$

($|x| \sim 2^j$ means $2^j < |x| < 2^{j+1}$). But a dilation argument shows that under the assumptions of Theorem 2

$$\iint_{\substack{|x|\geqslant 2^j \\ |y|\geqslant 2^l}} |\Delta^{1,2}_{h,k}(K)|^2 (x, y)\, dxdy = O(2^{-j(n+\alpha)} 2^{-l(m+\alpha)})$$

so that

$$(*) \leqslant \sum_{l\geqslant 0} \sum_{j\geqslant 0} 2^{-j(\alpha-\epsilon)} 2^{-l(\alpha-\epsilon)} < \infty$$

if ϵ is chosen $< \alpha$.

Finally, we indicate how to prove Theorem 3. The main idea is that under the assumptions on K of Theorem 3, the kernels $x(\partial K/\partial x)$ and $y(\partial K/\partial y)$ satisfy the same properties as K does. But when we may apply Theorem 1 to conclude that if $m(\xi, \eta) = \hat{K}(\xi, \eta)$ then $m(\xi, \eta) \in L^\infty$, $\xi(\partial m/\partial \xi)(\xi, \eta) \in L^\infty$ and $\eta(\partial m/\partial \eta)(\xi, \eta) \in L^\infty$.

Then we may apply the Marcinkiewicz Multiplier Theorem to conclude that T is bounded on L^p, $1 < p < \infty$. Together with Stein we have also shown that T maps $L \log^+ L$ to weak L^1. To see this according to Gundy-Stein [4] $S(Tf) \leqslant Cg^*_\lambda(f)$ for $\lambda > 1$. If $f \in L \log^+ L$ then $g_\lambda(f) \in$ weak L^1 for λ large enough. So $S(Tf) \in$ weak L^1. Modulo certain simple differences between probabilistic and nonprobabilistic area functions (see [4]) we see that

$$m\{(Tf)^* > \alpha\} \leqslant (\sim) \frac{1}{\alpha^2} \int_{S(Tf)\leqslant c\alpha} S^2(Tf) dx + m\{S(Tf) > c\alpha\}$$

where $(Tf)^*$ denotes the nontangential maximal function of Tf. So

$$(\sim) \leqslant \frac{1}{\alpha^2}\int_0^{c\alpha} m\{S(Tf) > \lambda\} \cdot \lambda d\lambda + m\{S(Tf) > c\alpha\}$$

$$\leqslant \frac{1}{\alpha^2}\int_0^{c\alpha} \frac{C'}{\lambda} \cdot \lambda d\lambda + \frac{C'}{c\alpha} \leqslant \frac{C''}{\alpha}.$$

So $Tf \in$ Weak L^1.

BIBLIOGRAPHY

1. A. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952).

2. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

3. A. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Adv. in Math. 16 (1975).

4. R. Gundy and E. M. Stein, *H^p theory for the poly-disc*, Proc. Nat. Acad. Sci. 76, No. 3 (1979).

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