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ON A CONJECTURE OF PAPAKYRIAKOPOULOS

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ABSTRACT. We disprove a conjecture of Swarup which in turn disproves a well-known conjecture of Papakyriakopoulos that a certain cover is planar.

Let

$$K_n = \left\langle a_1, b_1, \ldots, a_n, b_n; \prod_{i=1}^{n} (a_i, b_i) \right\rangle$$

and

$$J_n = \left\langle a_1, b_1, \ldots, a_n, b_n; \prod_{i=1}^{n} (a_i, b_i), (a_1, b_1\tau) \right\rangle,$$

where n is a fixed integer $\geqslant 2$ and τ is an element of the commutator subgroup of the free group $F(\{a_1, b_1, \ldots, a_n, b_n\})$. Further let S_n be the orientable closed surface of genus n. The fundamental group of S_n is K_n. Papakyriakopoulos [3] put forward the following

P.1. CONJECTURE. (a) J_n is torsion free and

(b) the cover of S_n corresponding to the kernel of the natural group homomorphism $K_n \longrightarrow J_n$ is planar.

Papakyriakopoulos [3] showed that if P.1. is true, then so is the Poincaré Conjecture.

G. A. Swarup [5] has posed the following

P.2. CONJECTURE. The group J_n is a nontrivial free product.

G. A. Swarup [5] showed that

$$\text{P.1.} \Rightarrow \text{P.2.} \Rightarrow \text{Poincaré Conjecture.}$$

THEOREM. *The conjecture* P.2. *is not in general true. Hence the conjecture* P.1. *is not in general true.*

PROOF. Let $G_1 = \langle a_1, b_1; (a_1, b_1c)\rangle$, where c is any fixed element of the commutator subgroup $F(\{a_1, b_1\})'$ of the free group $F(\{a_1, b_1\})$ so that (a_1, b_1c) is not conjugate to $(a_1, b_1)^{\pm 1}$ in $F(\{a_1, b_1\})$. For example one could take

$$c = (a_1, b_1).$$

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Take

$$G_2 = \langle a_2, b_2, \ldots, a_n, b_n; - \rangle$$

and $G = G_1 *_H G_2$, where $H = \langle h; - \rangle$ and the amalgamating isomorphisms are given by

$$\varphi_1(h) = (a_1, b_1) \quad \text{and} \quad \varphi_2(h) = (a_n, b_n)^{-1} \cdots (a_2, b_2)^{-1}.$$

Now G is an example of a group of the type denoted by J_n above. This is so since G_1 is torsion free by Magnus, Karrass and Solitar [2, §4.4, Theorem 4.12]. Also $(a_1, b_1) \neq e$ in G_1, because of the assumption on c and Magnus, Karrass and Solitar [2, §4.4, Theorem 4.11]. We show below that G cannot be decomposed into a proper free product.

Suppose that contrary to the above assertion we have that $G = X * Y$, where X and Y are nontrivial groups. The rank of G is $2n$, since G/G' is a free abelian group of rank $2n$. The rank of G_1 is 2 and the group G_1 is torsion free by Magnus, Karrass and Solitar [2, §4.4, Theorem 4.12]. Hence if G_1 is a proper free product, then G_1 is a free group of rank 2, by Theorem of B. H. Neumann on the rank of a free product (see for instance Magnus, Karrass and Solitar [2, §4.1, p. 192]. However the group G_1 is (by definition) clearly not a free group. So G_1 cannot be decomposed into a proper free product. Hence, by the Kuroš Subgroup Theorem for a free product, it follows that

$$(*) \qquad G_1 \subseteq g^{-1}Xg \quad \text{for some element } g \text{ of } G.$$

Hence

$$G/\overline{G_1^G} \cong (X/\overline{gG_1g^{-1}}^X) * Y.$$

Also

$$G/\overline{G_1^G} \cong G_2/\overline{\langle\varphi_2(h)\rangle}^{G_2} \quad \text{is a surface group,}$$

since G has generators $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ and defining relations

$$\prod_{i=1}^{n} (a_i, b_i) = e \quad \text{and} \quad (a_1, b_1c) = e.$$

Now a result of A. Shenitzer (see Proposition 5.14 of Lyndon and Schupp [1, Chapter II]) tells us that $G/\overline{G_1^G}$ cannot be both a surface group and a proper free product. Hence

$$X = \overline{gG_1g^{-1}}^X \quad \text{and } Y \cong G_2/\overline{\langle\varphi_2(h)\rangle}^{G_2}.$$

Thus the rank of Y is $2n - 2 \geqslant 2$.

All conjugates of Y intersect the subgroup H (of G) trivially. For

$$X \cap \overline{Y^G} = e \quad \text{and} \quad X = \overline{gG_1g^{-1}}^X \supseteq gHg^{-1}.$$

Hence, by the Subgroup Theorem of H. Neumann for free products with amalgamation (see for instance Lyndon and Schupp [1, Chapter IV, Theorem 6.6]), the group Y is either a proper free product or is contained in some conjugate of one of the groups G_1 and G_2. None of these possibilities can in fact occur.

(i) Y cannot be a proper free product, by the above-mentioned result of A. Shenitzer, since it is a surface group.

(ii) Y cannot be contained in a conjugate of G_1, since this would imply by (*) that Y is conjugate to a subgroup of X.

(iii) Y cannot be contained in a conjugate of G_2, since if it were Y would be a free group (as G_2 is a free group) which is false (a surface group is not a free group).

REMARK. As is well-known E. S. Rapaport [4] established Conjecture P.1. (a). Hence we have shown that Conjecture P.1. (b) does not in general hold.

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