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BOOK REVIEWS

Hilbert's fourth problem, by A. V. Pogorelov, Wiley, New York; Holt, Rinehart and Winston, New York, 1979, vi + 97 pp., $16.00.

Hilbert's Problem 4 stripped of its comments[1] is this: Omit from his axioms for the foundations of geometry besides the parallel axioms all those which contain the concept of angle, and replace them by the triangle inequality, which follows from the congruence axiom for triangles (CT).

(1) Determine all geometries satisfying these conditions.

(2) Study the individual ones.

This is not quite Hilbert's formulation because with respect to angles he only omits the CT. The remaining angle axioms have no significant applications without CT. Pogorelov uses the preferable form given above.

Only two such geometries, besides the elementary ones, were known at the time, the Minkowskian satisfying the euclidean parallel axiom and Hilbert's geometry generalizing in a similar way the hyperbolic situation. It seems that Hilbert did not think of a mixed situation like a half plane.

Nor is it clear, whether he wanted to include nonsymmetric (n.s.) distances (a Minkowski metric may be symmetric or not). Hilbert was admittedly probing and considered, in fact, an analysis of the distance concept as one of the tasks connected with the problem. Since absence of symmetry violates one of the axioms Pogorelov does not admit n.s. distances.

That Hilbert was interested in them is evident from Hamel's thesis **[2]**, which he directed immediately after his lecture. It dwells largely on n.s. distances. They were deemphasized in the later version **[3]** of **[2]**, most probably because from the great variety of Desarguesian metrics (both symmetric and not) which Hamel exhibited, no appealing new one emerged. The situation changed in 1929, when Funk discovered a very interesting, always n.s. geometry which resembles euclidean geometry in some respects and hyperbolic in others. In addition it led to the proper definition of completeness: The balls $\{x|px \leqslant \rho\}$ are compact, but not necessarily the $\{x|xp \leqslant \rho\}$, where xy is the distance.

Hamel approached the problem through the Weierstrass Theory of the calculus of variations, which requires smoothness properties alien to the foundations of geometry, the framework visualized by Hilbert for the problem. This is less surprizing than it appears because Hilbert was very interested

[1]These are translated in *Hilbert's mathematical problems* **8** (1902) of this Bulletin. The translation is reprinted in **[1]**, a report on the symposium on Hilbert's Problems held in 1974. Problem 4 appears there on pp. 131–141 with the title *Desarguesian spaces*. Hilbert's heading, slightly modernized, is: *The geometries in which the ordinary lines provide the shortest connections.* We refer the reader to the article in **[1]** for all facts concerning Problem 4 omitted here, because they are not connected to Pogorelov's book.

in the Weierstrass Theory at the time; in fact, he contributed his important invariant integral in his comments to Problem 23.

The discovery of the great variety of solutions showed that part (2) of Problem 4 is not feasible. *It is therefore no longer considered as part of the problem*. But many interesting special cases have been studied since 1929.

[**2**] remains to this day the only attack on the n.s. case. Therefore I only give the modern version of Problem 4 for symmetric distances. The length of a curve in a metric space is defined in strict analogy to the elementary procedure. The *Desarguesian* spaces R are given by the following conditions.

(1) *R is a nonempty open subset of the real projective space P^n metrized by a distance xy* (providing the correct topology) such that

(2) *The balls $\{x \mid px \leqslant \rho\}$ are compact* (or the Bolzano-Weierstrass Theorem holds).

(3) *Any two distinct points x, y can be connected by a segment, i.e. a curve of length xy.*

(4) *$xy + yz > xz$ when x, y, z are not collinear in P^n.*

In the spirit of the time Hilbert restricted himself to $n = 2, 3$ and so does Pogorelev. However, this has doubtless pedagogical reasons, because he addresses a wide class of readers. The real difference is between $n = 2$ and $n > 2$. Pogorelev's method works for $n > 3$, but requires greater technicalities. For the same reason he *assumes* that R is either all of $P^{2,3}$ or all of the affine $A^{2,3}$ or a convex subset of $A^{2,3}$, although Hamel *proved* in [**3**] that R must have one of these forms (this holds actually for all n and even n.s. distances). The proof becomes simple by using covering spaces, which Pogorelev avoids. Thus he accomplishes more than appears at first sight.

I will explain in some detail his principal contribution, because for my report in [**1**] the Russian book (price: 24 kopeks) was not yet at my disposal.

Led by integral geometry I had observed (references are found in the translation) that many Desarguesian spaces can be obtained as follows, taking $R = P^n$ as an example. Denote hyperplanes by H, and for a point set M put $H_M = \{H \mid H \cap M \neq \varnothing\}$. Define on the set of all hyperplanes a nonnegative measure ε with (a) $\varepsilon(H_{\{p\}}) = 0$ for each point p, (b) $\varepsilon(H_M) > 0$ when $M \neq \varnothing$ is open, (c) $\varepsilon(H_{P^n}) = 2k < \infty$. Notice that $\varepsilon(H_L) = 2k$ for any line L because $H_L = H_{P^n}$. Two distinct points x, y divide the line L through them into two arcs A_1, A_2 and $\varepsilon(H_{A_1}) + \varepsilon(H_{A_2}) = \varepsilon(H_L) = 2k$ because of (a), so that at least one A_i, say A_1, satisfies $\varepsilon(H_{A_1}) \leqslant k$. Putting $xy = \varepsilon(H_{A_1})$ defines a Desarguesian metrization of P^n with A_1 as segment.

Without essential changes this construction can be used when $R \subset A^n$ by admitting the value ∞ for $\varepsilon(H_M)$.

Pogorelev proves that in detail. Now the question arises whether this method yields all Desarguesian metrics. *The answer is affirmative for $n = 2$, but $n > 2$ requires measures δ, which may take negative values on many sets.* Unexpectedly, this proves compatible with $\delta(H_T) > 0$ for every segment T. I had noticed this situation previously in the very special case of the Minkowskian geometries.

We denote a metric given by a δ with $\delta(H_T) > 0$ as a δ-metric. *The* *construction of δ (or ε for $n = 2$) for a given Desarguesian metric is Pogorelev's prime contribution.*

He shows first that every δ-metric can be approximated, uniformly on every compact set, with Desarguesian metrics derived from integrands $F(x, dx)$ satisfying the standard conditions of the calculus of variations $(n = 2, 3)$.

In cartesian coordinates the euclidean densities d_eL, d_eH (i.e. $H_e(M) = \int_{H \cap M \neq \varnothing} d_eH$) for a line L in P^2 or a plane H in normal form $x \cdot u - p = 0$, is $d_eL = dudp$, $d_eH = dudp$ (u is a unit vector normal to L, resp. H, and p the distance from the origin). It is clear that in the general case the densities must have the form $\gamma(L)d_eL$ or $\gamma(H)d_eH$. *Thus the problem becomes finding γ in terms of $F(x, dx)$.*

By an ingenious use of a special form which the Euler equations take when the straight lines are the extremals, this task is reduced to Funk's problem of representing the supporting function $G(y)$ of a convex body K with the origin as center in the form $G(y) = \int_U \gamma(u)|y \cdot u|du$ with even $\gamma(u)$. The problem is easily solved with $\gamma(u) \geqslant 0$ for $n = 2$, but is far from trivial for $n = 3$ and $\gamma(u)$ may take negative values.[2]

The solution is applied for each x to $F(x, \dot{x})$ with $y = \dot{x}$ in $\dot{x}$-space. $F(x, y)$ is, in fact, because of $F(x, kx) = |k|F(x, x)$ and the Legendre condition, the supporting function of a convex body with the origin as center in y-space. This defines $\gamma(u, x \cdot u)$ and $dH = \gamma(u, xu)dpdu$.

The final answer for $\delta(n = 3)$ is expressed in terms of the measure $\delta'(Q')$ of the point set Q' obtained by a correlation from a set Q of planes where $\delta'(Q') = \delta(Q)$. *The conditions are $\delta'(P) = 0$ for every plane P* (corresponding to (a)), *$\delta'(C) \geqslant 0$ for every cone C and $\delta'(C) > 0$ when C has interior points.*

The very well written book begins with an introduction to the most elementary facts concerning $P^{2,3}$ whose points are defined as equivalence classes of triples, resp. quadruples of reals. It then gives topologies for the lines and planes. The detailed exposition of the results sketched above follows. Finally the tract turns to the foundations of geometry (with Pogorelev's own variation of Hilbert's axioms) where it is clearly undesirable to take the analytic definition of $P^{2,3}$ for granted. The special geometries of Minkowski and Hilbert are discussed. As all this is rather known territory, I will not dwell on it.

The blurb on the cover states that the book is accessible to advanced undergraduates. This seems unlikely if for no other reason than the breadth of knowledge which is taken for granted and is certainly so in the Russian original, which provides practically no references. The excellent translation by R. A. Silverman contains a good list of these, for all knowledge assumed, and a very judicious four pages of notes, both contributed by E. Zaustinsky, which will greatly help the reader (not only students) to locate the material,

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[2]For readers familiar with the theory of convex bodies I mention that $\gamma(u) > 0$ means that K is a projection body, so that the δ-metric is an ε-metric, when $F(x, y)$ supports for each fixed x a projection body in y-space (see below). It is trivial that a two-dimensional K is always a projection body, so that γ is easy to find, but for $n > 2$ projection bodies are very special cases.

and beyond this explains at crucial points the reasons for certain steps which could baffle a newcomer to the field.

References

1. Felix E. Browder (ed.), *Mathematical developments arising from Hilbert problems*, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R. I., 1976.

2. G. Hamel, *Ueber die Geometrien, in denen die Geraden die Kürzesten sind*, Dissertation, Göttingen, 1901, 90 pp.

3. ______, same title, Math. Ann. **57** (1903), 231–264.

HERBERT BUSEMANN

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Transformation groups and representation theory, by Tammo tom Dieck, Lecture Notes in Math., vol. 766, Springer-Verlag, Berlin and New York, 1979, viii + 300 pp., $18.00.

Let G be a topological group and X a topological space. An action of G on X is a continuous map $G \times X \to X$, written $(g, x) \to gx$ on elements, such that $1x = x$ and $g(g'x) = (gg')x$. The study of such group actions is a major and growing branch of topology.

Probably the longest established aspect of this study concerns smooth actions of compact Lie groups on differentiable manifolds. Typically, one tries to classify such actions on a given manifold or to construct particularly nice or particularly pathological examples. A recent concern, still very much in its infancy, is the analysis of the algebraic topology of G-spaces. This book is largely concerned with aspects of this new subject of equivariant homotopy theory.

While some formal theory goes through more generally, it is widely accepted that the appropriate level of generality is to restrict attention to compact Lie groups. Here there is a dichotomy. Many parts of the theory become very much simpler when one restricts further to finite groups, but one feels that one really doesn't understand the theory unless one can carry it out for all compact Lie groups.

The major computable invariants of algebraic topology are "stable". That is, with a shift of indexing, they are the same for a based space X and for its suspensions $\Sigma^n X = X \wedge S^n$. Here the smash product $X \wedge Y$ is the quotient of $X \times Y$ by the wedge, or 1-point union, $X \vee Y$. In equivariant algebraic topology, this description will not do. It makes little sense to restrict attention to spheres with trivial G-action. Since it would be unmanageable to allow spheres with arbitrary G-action, it is best to understand G-spheres to be 1-point compactifications SV of representations V. Here V is a finite-dimensional real inner product space with G acting through isometries. With basepoints fixed under the action of G, "stable" invariants of based G-spaces should be the same for X and for $\Sigma^V X = X \wedge SV$, where G acts diagonally