BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 2, September 1980

SK_1 OF p-ADIC GROUP RINGS

BY ROBERT OLIVER

If A is a Dedekind domain with quotient field K, and π a finite group, define

$$SK_1(A\pi) = \mathrm{Ker}[K_1(A\pi) \longrightarrow K_1(K\pi)].$$

We concentrate here on the case when A is a *p-ring*—the ring of integers in a finite extension of the p-adic rationals $\hat{\mathbf{Q}}_p$—and report on results which completely calculate $SK_1(A\pi)$ in this case.

The main reason for looking at $SK_1(A\pi)$ involves $SK_1(\mathbf{Z}\pi)$, shown by Wall [5] to be the torsion subgroup of the Whitehead group $\mathrm{Wh}(\pi)$ (and thus having various topological applications). The inclusions $\mathbf{Z}\pi \subseteq \hat{\mathbf{Z}}_p[\pi]$ induce a surjection

$$SK_1(\mathbf{Z}\pi) \twoheadrightarrow \sum_p SK_1(\hat{\mathbf{Z}}_p[\pi])$$

(see §1 in [3]), whose kernel is denoted $\mathrm{Cl}_1(\mathbf{Z}\pi)$. The computation of $SK_1(\mathbf{Z}\pi)$ thus splits into two parts. $\mathrm{Cl}_1(\mathbf{Z}\pi)$ can be calculated in many cases (see, e.g., [4] and [3], noting that $\mathrm{Cl}_1(\mathbf{Z}\pi) = SK_1(\mathbf{Z}\pi)$ for abelian π); but no general formula or algorithm has yet been found. The groups $SK_1(\hat{\mathbf{Z}}_p[\pi])$, on the other hand, are completely described by Theorems 1 and 2 below.

For any finite π, define

$$H_2^{ab}(\pi) = \mathrm{Im}[\textstyle\sum\{H_2(\rho): \rho \subseteq \pi, \rho \text{ abelian}\} \longrightarrow H_2(\pi)].$$

If π is a p-group, the situation is particularly simple.

THEOREM 1. *For any p-ring A and p-group* π,

$$SK_1(A\pi) \cong H_2(\pi)/H_2^{ab}(\pi).$$

Note in particular that $SK_1(A\pi)$ is independent of A in this case. If $B \supseteq A$ is a totally ramified extension of p-rings, the inclusion $A\pi \subseteq B\pi$ induces an isomorphism from $SK_1(A\pi)$ to $SK_1(B\pi)$. If, on the other hand, $B \supseteq A$ is an unramified extension, it is the transfer map

$$trf\colon SK_1(B\pi) \longrightarrow SK_1(A\pi)$$

which is an isomorphism.

For arbitrary finite π, the formula is much messier. For any p-ring A and finite group π, set $n = \exp(\pi)$ and regard $\mathrm{Gal}(A\zeta_n/A)$ (ζ_n a primitive nth root of

Received by the editors April 7, 1980.
1980 *Mathematics Subject Classification.* Primary 18F25; Secondary 16A26.

unity) as a subgroup of $(\mathbf{Z}_n)^*$. Two elements $g, h \in \pi$ will be called *A-conjugate* if $g^a = xhx^{-1}$ for some $x \in \pi$ and $a \in \mathrm{Gal}(A\zeta_n/A)$.

THEOREM 2. *Let A be a p-ring, π a finite group, and set $n = \exp(\pi)$. Let $g_1, \ldots, g_k$ be A-conjugacy class representatives for elements of order prime to p. Define, for $1 \leqslant i \leqslant k$,*

$$Z_i = Z_\pi(g_i); \quad N_i = \{x \in \pi: xg_ix^{-1} = g_i^a \text{ for some } a \in \mathrm{Gal}(A\zeta_n/A)\}.$$

Then

$$SK_1(A\pi) \cong \sum_{i=1}^{k} H_0(N_i; H_2(Z_i)/H_2^{ab}(Z_i))_{(p)}.$$

Assume again that π is a p-group. Just finding a map between $SK_1(A\pi)$ and $H_2(\pi)/H_2^{ab}(\pi)$ takes a fair amount of machinery. For simplicity, assume A is unramified over $\hat{\mathbf{Z}}_p$ (ramified p-rings must be dealt with separately). A short exact sequence

$$0 \longrightarrow \mathrm{Wh}'(A\pi) \xrightarrow{\Gamma} \overline{I(A\pi)} \xrightarrow{\omega} \pi^{ab} \longrightarrow 0 \tag{1}$$

is first constructed where

$$\mathrm{Wh}'(A\pi) = K_1(A\pi)/(A^* \times \pi^{ab} \times SK_1(A\pi))$$

$$I(A\pi) = \mathrm{Ker}(A\pi \longrightarrow A), \quad \overline{I(A\pi)} = I(A\pi)/\langle x - gxg^{-1}: x \in I(A\pi), g \in \pi\rangle,$$

$$\omega(\textstyle\sum r_ig_i) = \prod [g_i]^{\mathrm{Tr}(r_i)} \qquad (\mathrm{Tr}: A \longrightarrow \hat{\mathbf{Z}}_p \text{ the trace map})$$

and Γ is defined using the p-adic logarithm.

The sequence (1) gives no new information about $\mathrm{Wh}'(A\pi)$ as an abstract group, but it does allow more control over it. For example, when π is a 2-group and $\mathrm{Wh}'(\hat{\mathbf{Z}}_2[\pi])$ has the involution induced by $g \longrightarrow g^{-1}$, (1) yields the simple formula

$$H^1(\mathbf{Z}_2; \mathrm{Wh}'(\hat{\mathbf{Z}}_2[\pi])) \cong \frac{\{[g] \in \pi^{ab}: [g^2] = e \text{ in } \pi^{ab}\}}{\langle [g] \in \pi^{ab}: g \text{ conjugate to } g^{-1}\rangle}$$

(answering a question of Wall).

Given any extension $1 \longrightarrow \rho \longrightarrow \tilde{\pi} \longrightarrow \pi \longrightarrow 1$ of p-groups, (1) is used to get a diagram (with exact rows)

$$\begin{array}{ccccccccc}
0 \longrightarrow & SK_1(A\rho) & \longrightarrow & \mathrm{Wh}(A\rho) & \xrightarrow{\Gamma} & \overline{I(A\rho)} & \xrightarrow{\omega} & \rho^{ab} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & SK_1(A\tilde{\pi}) & \longrightarrow & \mathrm{Wh}(A\tilde{\pi}) & \xrightarrow{\Gamma} & \overline{I(A\tilde{\pi})} & \xrightarrow{\omega} & \tilde{\pi}^{ab} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & SK_1(A\pi) & \longrightarrow & \mathrm{Wh}(A\pi) & \xrightarrow{\Gamma} & \overline{I(A\pi)} & \xrightarrow{\omega} & \pi^{ab} & \longrightarrow 0.
\end{array}$$

A little diagram chasing yields a homomorphism

$$\Delta\colon\ \mathrm{Ker}[\rho^{ab} \longrightarrow \tilde{\pi}^{ab}] \longrightarrow \mathrm{Coker}[SK_1(A\tilde{\pi}) \longrightarrow SK_1(A\pi)],$$

whose kernel is easily seen to contain $[\rho, \tilde{\pi}]$. The spectral sequence for the extension induces an exact sequence

$$H_2(\pi) \xrightarrow{\delta} \rho/[\rho, \tilde{\pi}] \longrightarrow \tilde{\pi}^{ab}$$

and the composite $\Delta \circ \delta$ is a natural homomorphism

$$H_2(\pi) \longrightarrow \mathrm{Coker}[SK_1(A\tilde{\pi}) \longrightarrow SK_1(A\pi)].$$

That this induces an isomorphism $H_2(\pi)/H_2^{ab}(\pi) \cong SK_1(A\pi)$ (for proper choice of $\tilde{\pi}$) now follows upon checking:

(A) Δ induces an isomorphism $\rho_0/\rho_1 \cong \mathrm{Coker}[SK_1(A\tilde{\pi}) \longrightarrow SK_1(A\pi)]$, where

$$\rho_0 = \rho \cap [\tilde{\pi}, \tilde{\pi}] \quad \text{and} \quad \rho_1 = \langle z \in \rho\colon z = ghg^{-1}h^{-1} \text{ for some } g, h \in \tilde{\pi}\rangle.$$

(B) $\rho_0 = \delta(H_2(\pi))$, $\rho_1 = \delta(H_2^{ab}(\pi))$, and for $\tilde{\pi}$ large enough, δ induces an isomorphism $H_2(\pi)/H_2^{ab}(\pi) \cong \rho_0/\rho_1$.

(C) There exists $\tilde{\pi} \twoheadrightarrow \pi$ such that $SK_1(A\tilde{\pi}) = 0$.

In other words, it is the difference between "actual" commutators and elements in $[\tilde{\pi}, \tilde{\pi}]$ which gives rise to elements in $SK_1(A\pi)$. Note that by (A), and the surjectivity of the localization map, surjections $\tilde{\pi} \twoheadrightarrow \pi$ of p-groups can be constructed such that the induced map

$$SK_1(\mathbf{Z}\tilde{\pi}) \longrightarrow SK_1(\mathbf{Z}\pi)$$

is *not* onto.

Once $SK_1(A\pi)$ is computed for p-groups π, the result for general π is obtained by first extending to certain twisted group rings, and then applying the induction theory in [1]. In particular, one gets in the process (using also Theorem 1 in [3])

THEOREM 3. *$SK_1(A\pi)$ (A any p-ring) and $SK_1(\mathbf{Z}\pi)_{(p)}$ are generated by induction from p-elementary subgroups of π.*

As examples of specific computations, we get

THEOREM 4. *Let A be a p-ring and π a finite group. Then $SK_1(A\pi) = 0$ if* (i) *π_p (p-Sylow subgroup) has a normal abelian subgroup with cyclic quotient or* (ii) *π is a symmetric or alternating group.*

Specific examples of p-groups π with $H_2(\pi)/H_2^{ab}(\pi) \neq 0$ are constructed in [2]. The smallest such π occur when $|\pi| = p^5$ (p odd) or $|\pi| = 64$.

Combining these results with those on $\mathrm{Cl}_1(\mathbf{Z}\pi)$ in [3], we get

THEOREM 5. *$SK_1(Z\pi) = 0$ if π is any symmetric group or generalized quaternionic group, or if $\pi \cong SL(2, p)$ or $PSL(2, p)$ for p prime. In particular,* $\mathrm{Wh}(\Sigma_n) = 0$.

REFERENCES

1. A. Dress, *Induction and structure theorems for orthogonal representations of finite groups,* Ann. of Math. (2) **102** (1975), 291–325.

2. W. Neumann, *Manifold cutting and pasting groups,* Topology **15** (1975), 237–244.

3. R. Oliver, *SK_1 for finite group rings.* I, Invent. Math. **57** (1980), 183–204.

4. M. Stein, *Whitehead groups of finite groups,* Bull. Amer. Math. Soc. **84** (1978), 201–212.

5. C. T. C. Wall, *Norms of units in group rings,* Proc. London Math. Soc. **29** (1974), 593–632.

MATEMATISK INSTITUT, AARHUS UNIVERSITET, 8000 AARHUS C, DENMARK