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SPHERICAL FIBRATIONS

BY J. L. NOAKES

ABSTRACT. In [8], [9] we extend some theorems of I. M. James and J. H. C. Whitehead on the homotopy type of spherical fibrations. Here we sketch our results and methods.

1. Introduction. Let E_1, E_2 be q-sphere Hurewicz fiberings ($q \geqslant 1$) over the same connected finite *CW*-complex B. We suppose that the spaces E_1, E_2 have the same homotopy type ($E_1 \simeq E_2$), that E_2 has a cross-section, and that E_2 is orientable. Then by [9, Theorem 1] E_1 is also orientable. We suppose that B is *nilpotent* [1], and let $E_{j(p)} \longrightarrow B_{(p)}$ be the localization of $E_j \longrightarrow B$ at a prime p ($j = 1, 2$).

THEOREM 1. *If* $\dim B < 2q$ *then, for any prime* p, $E_{1(p)}$ *has a cross-section.*

In [3] I. M. James and J. H. C. Whitehead prove a similar result for the case where B is a sphere, but where the fibre of E_1, E_2 is not necessarily a sphere. We comment on our proof in §2.

THEOREM 2. *If* $E_1 \simeq B \times S^q$ *where* E_1 *has a cross-section then* E_1 *is fibre homotopy trivial.*

In [4] James and Whitehead take B to be a sphere and prove a similar result. Our proof in [9] uses a counting argument in the spirit of [3], [7]. Comparing this proof with Theorem 1 we find that if $E_1 \simeq B \times S^q$ where $\dim B < 2q$ then E_1 is fibre homotopy trivial. It was a conjecture along these lines by I. M. James that led me to write [8], [9]. I wish to thank Professor James for telling me his conjecture.

Recall that the *fibre suspension* [6] $\Sigma E \longrightarrow B$ of a map $\pi : E \longrightarrow B$ is defined as follows. Let ΣE be the quotient of $E \times [-1, 1]$ by the relations $(e, -1) \sim (e', -1)$ and $(e, 1) \sim (e', 1)$ for $\pi(e) = \pi(e')$. Then the projection of ΣE takes $[e, t]$ to $\pi(e)$. We assume that E_2 has the fibre homotopy type of ΣE for some E.

REMARKS. (i) If $\dim B < q$ then this assumption holds automatically.

(ii) If E_2 is a fibre bundle then this assumption is equivalent to the requirement that E_2 have a cross-section.

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THEOREM 3. *If either $H^q(B; Z)$ is finite or $\pi_q B = 0$ then there is a homotopy equivalence $f: B \longrightarrow B$ such that E_1 has the fibre homotopy type of the induced fibration $f^* E_2$.*

EXAMPLE. Let $\Phi \subseteq \pi_{r+q} S^q$ $(0 < r < q - 1)$ be maximal with respect to the property that if $\alpha, \beta \in \Phi$ satisfy $\alpha + \beta = 0$ then $\alpha = \beta$. It follows from Theorem 3 that there is a bijection from Φ onto the set of homotopy types of q-sphere fibrations over S^{r+1}.

Let S_j^q be the fibre of E_j $(j = 1, 2)$.

THEOREM 4. *If the pairs of spaces (E_1, S_1^q), (E_2, S_2^q) have the same homotopy type then there is a homotopy equivalence $f: B \longrightarrow B$ such that E_1 has the fibre homotopy type of $f^* E_2$.*

In [4] James and Whitehead prove a similar result for the case where B is a sphere. In [2] S. Y. Husseini considers the situation of Theorem 4, with the additional hypotheses that $\dim B < q$ and $\pi_1 B = 0$. Simple examples show that [2, Theorem 1.1] is false as stated.

2. Methods. Our proofs of Theorems 3 and 4 turn on an elementary construction which is difficult to describe more briefly than in [9]. However, it may be helpful to say what this construction aims to do.

Let $\pi_j: E_j \longrightarrow B$ $(j = 1, 2)$ be the projections, let s_2 be a cross-section of E_2 and let $\gamma: E_2 \longrightarrow E_1$ be a homotopy equivalence. (In the proof of Theorem 4 we take γ to be a map of pairs.) A calculation shows that $\pi_1 \gamma s_2$ is a homotopy equivalence, and our construction aims to replace γ by a homotopy equivalence θ' so that the diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\theta'} & E_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ B & \xrightarrow[\pi_1 \gamma s_2]{} & B \end{array}$$

commutes.

The proof of Theorem 1 is easier to summarise. A counting argument taken from [7] shows that the Gysin sequences of the $E_{j(p)}$ split

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^rB_{(p)} & \xrightarrow{\pi_1^*} & H^rE_{1(p)} & \xrightarrow{\psi_1} & H^{r-q}B_{(p)} & \longrightarrow & 0 \\ & & & & \gamma^* \downarrow\uparrow \tau^* & & & & \\ 0 & \longrightarrow & H^rB_{(p)} & \xrightarrow{\pi_2^*} & H^rE_{2(p)} & \xrightarrow{\psi_2} & H^{r-q}B_{(p)} & \longrightarrow & 0 \end{array}$$

for all r. Here τ is a homotopy inverse of γ, and we take coefficients in the ring $\mathbf{Z}_{(p)}$ of integers localized at the prime p. Our notation does not distinguish a map from its localization.

We choose $a_0 \in H^qE_{2(p)}$ so that $\psi_2 a_0$ is the identity of $H^0B_{(p)}$. Let $a_2 = a_0 - \pi_2^* s_2^* a_0$. Then we consider two cases separately.

(1) *There is no exchange at p* when $\psi_1 \tau^* a_2$ is a unit of $H^0B_{(p)}$.

(2) *There is an exchange at p* when $\psi_1 \tau^* a_2$ is not a unit of $H^0B_{(p)}$.

When there is no exchange a computation shows that $\pi_1 \gamma s_2 : B \longrightarrow B$ is a homotopy equivalence at the prime p. It quickly follows that $E_{1(p)}$ has a cross-section.

When there is an exchange we form the induced fibrations $\pi_1^* E_{1(p)}, E = \iota^* \gamma^* \pi_1^* E_{1(p)}$ over $E_{1(p)}, E_{2(p)}|(B^{q-1})_{(p)}$. Here B^{q-1} is the $(q-1)$-skeleton of B, and ι is the inclusion of $E_{2(p)}|(B^{q-1})_{(p)}$ in $E_{2(p)}$.

$$\begin{array}{ccccc} E & \longrightarrow & \pi_1^*E_{1(p)} & \longrightarrow & E_{1(p)} \\ \downarrow & & \downarrow & & \downarrow \pi_1 \\ E_{2(p)}|(B^{q-1})_{(p)} & \xrightarrow[\gamma\iota]{} & E_{1(p)} & \xrightarrow[\pi_1]{} & B_{(p)} \end{array}$$

Then $\pi_1^* E_{1(p)}$ has a cross-section and therefore E has a cross-section.

Next a computation using (2) and $\dim B < 2q$ shows that $\pi_1 \gamma \iota$ is almost a homotopy equivalence. So the cross-section of E gives rise to a cross-section of $E_{1(p)}$. For details we refer to [8].

Theorems 1, 2, 3 and 4 contain as special cases all the results of [4] except the case $r = q$ of [4, Theorem 1.6]. This exception can be dealt with from our standpoint. There remains the question of what happens when neither E_1 nor E_2 has a cross-section: for background on this we refer to [5].

REFERENCES

1. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972.

2. S. Y. Husseini, *Spherical fibrations*, Lecture Notes in Math., vol. 249, Springer-Verlag, Berlin and New York, 1970, pp. 107–124.

3. I. M. James and J. H. C. Whitehead, *Note on fibre spaces*, Proc. London Math. Soc. **4** (1954), 129–137.

4. ———, *The homotopy theory of sphere bundles over spheres.* (I), Proc. London Math. Soc. 4 (1954), 196–218.

5. ———, *The homotopy theory of sphere bundles over sphere.* (II), Proc. Lon- Math. Soc. 5 (1955), 148–166.

6. I. M. James, *Overhomotopy theory*, Symposia Mathematica 4 (1970), 219–229.

7. ———, *Which fibre spaces are decomposable*?, Indag. Math. 37 (1975), 385–390.

8. J. L. Noakes, *Cross-sections and homotopy type* (*to appear*).

9. ———, *Fibrations and homotopy type* (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS, WESTERN AUSTRALIA 6009