AN IDENTITY WITH APPLICATIONS
TO HARMONIC MEASURE

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In this note we use an elementary integral formula to give a new short proof of the theorem of B. E. J. Dahlberg [2] on the mutual absolute continuity of harmonic and surface measure on bounded Lipschitz domains. (See [4] for the relevant definitions.) The formula also provides a new proof of the so-called reverse Holder inequality (see [2]), and L^p-estimates for the Dirichlet problem (see [3]). Furthermore, we are able to treat domains whose boundaries are worse than Lipschitz. In particular, we show that if $D \subset R^n$ is given locally (in the appropriate sense) by the graph of a function ϕ with $\nabla\phi \in L^p, p > n - 1$, and D is regular for the Dirichlet problem, then harmonic measure and surface measure on ∂D are mutually absolutely continuous. The hypothesis of regularity is superfluous when $n = 3$, or under the stronger assumption $\nabla\phi \in$ BMO, but not necessarily otherwise.

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THEOREM 1. *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$. Let ω be harmonic measure of ∂D with respect to $X_0 \in D$. Then*

(a) *ω and σ are mutually absolutely continuous.*

(b) *Let $k(Q) = d\omega/d\sigma$. Then $k \in L^2(d\sigma)$; moreover, for every surface ball $\Delta \subset \partial D$,*

$$(*) \qquad \left(\frac{1}{\sigma(\Delta)} \int_\Delta k^2(Q)\, d\sigma(Q) \right)^{1/2} \leqslant \frac{c}{\sigma(\Delta)} \int_\Delta k(Q) d\sigma(Q).$$

To establish our theorem, we need

LEMMA 1 (THE MAIN IDENTITY). *Assume that D is a bounded C^∞ domain in $\mathbf{R}^n$, containing 0. Let ω be harmonic measure of ∂D at 0. Then if $k = d\omega/d\sigma$,*

$$\frac{1}{\omega_n} \int_{\partial D} \frac{k(Q)}{|Q|^{n-2}}\, d\sigma(Q) = \int_{\partial D} k^2(Q) \langle Q, N_Q \rangle\, d\sigma(Q),$$

where N_Q is to outer unit normal, and ω_n the surface area of $S^{(n-1)}$.

PROOF. Let $G(X)$ be the Green function of D with pole at 0. Then

$$G(X) = 1/|X|^{n-2} - u(x),$$

where u is harmonic in D and G is 0 on ∂D. Moreover, $k(Q) = -(1/(n-2)\omega_n)(\partial G/\partial N_Q)(Q)$. Let $\alpha = Q - N_Q\langle Q, N_Q\rangle$. Because $\langle \alpha, N_Q\rangle = 0$, $\Sigma_{i=1}^n \alpha_i(Q)(\partial G/\partial x_i)(Q) = 0$. Thus $\int_{\partial D} k(Q)(\Sigma_{i=1}^n \alpha_i(Q)(\partial G/\partial x_i) d\sigma(Q)) = 0$. Hence

$$-\int_{\partial D} k(Q)\langle Q, \nabla G(Q)\rangle \, d\sigma(Q) = (n-2)\omega_n \int_{\partial D} k^2(Q)\langle Q, N_Q\rangle \, d\sigma(Q).$$

However, if $v(x) = x \nabla u(x)$, then v is harmonic in D and $v(0) = 0$. Thus $\int_{\partial D} k(Q)\langle Q, \nabla u(Q)\rangle d\sigma(Q) = 0$ and our identity follows.

By [4, p. 320], and the definition of Lipschitz domain, we note that it is enough to prove Theorem 1(a) for starshaped Lipschitz domains Ω, with the additional property that they can be exhausted by star-shaped C^∞ domains with comparable Lipschitz constants, and with boundaries in a 1-1 correspondence radially. Since Ω, Ω_j are regular for the Dirichlet problem, it is easy to see that if ω, ω_j are the respective harmonic measures, $\omega_j \longrightarrow \omega$ weakly. On the other hand, if we apply Lemma 1 to Ω_j, we see that (†) $\int_{\partial \Omega_j} k_j^2(Q) \leqslant c$ where c depends only on the Lipschitz constant of Ω. It then follows that there exists an $L^2(d\sigma)$ function k, satisfying (†) such that $d\omega = k d\sigma$. To prove that $\sigma \ll \omega$, we notice that once (†) is established, the elementary argument in [2, p. 280], can be repeated.

For the proof of Theorem 1(b), choose coordinates (x, y), $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, so that near a point Q_0 of ∂D, D is the region above the graph of a Lipschitz function ϕ and $\Delta = \Delta(Q_0, r) = \{(x, \phi(x)) \in \partial D/|x - x_0| < r\}$. Let A be a point above Δ, in D, with $\text{dist}(A, \Delta) \simeq Cr$. (*) follows from the estimates

(1) $\int_\Delta k_A^2 \, d\sigma(Q) \leqslant c/\sigma(\Delta)$,

(2) $k(Q) \leqslant ck_A(Q)\omega(\Delta)$ a.e. $Q \in \Delta$,

where $k_A(Q) = d\omega^A/d\sigma$. (2) follows from the estimate at the bottom of p. 512 in [5]. Estimate (1) has an easy proof when D is a special kind of domain. Suppose that D can be approximated in a regular way by C^∞ domains Ω_j and that

(3) $\text{dist}(A, \partial\Omega_j) \simeq r$.

(4) $\langle A - Q_j, N_{Q_j}\rangle \geqslant cr$, a.e. $Q_j \in \Delta_j$ (Δ_j corresponds to Δ).

(5) $\langle A - Q_j, N_{Q_j}\rangle \geqslant 0$, a.e. $Q_j \in \partial\Omega_j$ (Ω_j is star-shaped w.r.t. A).

The main identity, (3), (4) and (5) yield

$$\int_{\Delta_j} k_A^j(Q)^2 \, d\sigma_j(Q) \leqslant cr^{1-n},$$

and a passage to the limit gives (1).

(b) *Moreover, if $p > n - 1$, then $\sigma \ll \omega$.*

The proof of Theorem 3(a) is similar to that of Theorem 1(a). The only difference is that we apply Holder's inequality before invoking the main identity. For part (b) we use (see [6, Chapter 8]).

LEMMA. *If $\phi \in L_1^p(R^{n-1})$, $p > n - 1$, and ϕ is nonnegative and supported in a cube Q, then for any $\epsilon > 0$ there is a closed set F and a Lipschitz function ψ supported in Q such that $|Q - F| < \epsilon$, $\phi = \psi$ on F and $\phi \geqslant \psi \geqslant 0$.*

We will call a domain D in $\mathbf{R}^n$ *locally interior* (resp. *exterior*) *star-shaped* if there exist closed sets F_i and open sets V_i with $F_i \subset V_i$, $\partial D = \bigcup F_i$ and points $X_i \in \mathbf{R}^n$ and continuous functions ϕ_i on $S^{(n-1)}$, $0 < c \leqslant \phi_i$, such that in polar coordinates with X_i as origin,

$$D \cap V_i = \{(r, \theta) | \theta \in S^{n-1}, 0 \leqslant r < \phi_i(\theta)\} \cap V_i$$

and

$$F_i \subset \{(\phi_i(\theta), \theta) | \theta \in S^{n-1}\},$$

(resp. $\{(r, \theta) | 0 \leqslant r < \phi_i, \theta \in O_i$, a spherical cap in $S^{(n-1)}\} \subset {}^cD$ and $F_i \subset \{(\phi_i(\theta), \theta) | \theta \in O_i\}$).

COROLLARY. *Let D be a bounded domain, regular for the Dirichlet problem:*
(a) *If D is locally interior star-shaped and $\nabla\phi_i \in L^p(d\theta)$, $p > 2$, then $\omega \ll \sigma$. If $p > n - 1$, then $\sigma \ll \omega$.*
(b) *If D is locally exterior star-shaped, then in addition to (a) we have, $k \in L^q(d\sigma)$ for $q = 2(p - 1)/p$.*

The corollary follows easily from Theorem 3. To deduce part (b), we apply the Kelvin transform.

Some consequences of Wiener's criterion [7] are:
(1) When $n = 3$, the sole hypothesis $|\nabla\phi_i| \in L^p(d\theta)$ implies D is regular;
(2) When $n \geqslant 4$, a region with $|\nabla\phi| \in L^p(d\theta)$ for all $p < \infty$ may fail to be regular. However, when $\nabla\phi_i \in \text{BMO}$, D is regular since $\phi_i \in \Lambda_*$, the Zygmund class. Another sufficient condition for regularity is

$$\phi_j \in \text{Lip}(h(\log 1/h)^{1/(n-3)}).$$

The exponent $1/(n-3)$ is sharp. An example for which regularity fails is the domain below the graph of $\phi(x) = |x| (\log 1/|x|)^\alpha$, $x \in \mathbf{R}^{n-1}$, $\alpha > 1/(n-3)$. Note that $\nabla\phi \in L_{\text{loc}}^p(dx)$ for all $p < \infty$ and $\phi \in \text{Lip}(h(\log 1/h)^\alpha)$.

We can reduce the general case to this special case by replacing D with a larger, unbounded region D' with $\Delta \subset \partial D \cap \partial D'$. We then take the Kelvin transform $K(X) = \widetilde{X} = (X - R)/|X - R|^2$ with respect to a point $R \in E = {}^cD'$. Let $d\widetilde{\sigma}$ be surface measure on $K(\partial D')$ and let $\widetilde{k}_A(\widetilde{Q})d\widetilde{\sigma}(\widetilde{Q})$ be the harmonic measure at $\widetilde{A}$ for $K(D') \cup \{0\}$. If, $\operatorname{dist}(R, D') \geqslant c_0$, then $d\widetilde{\sigma}$ and $d\sigma$ are comparable. If $\widetilde{u}$ is harmonic in $K(D')$, then $|X - R|^{(2-n)}\widetilde{u}(\widetilde{X})$ is harmonic in D'. Therefore the maximum principle shows that $k_A(Q) \leqslant c\widetilde{k}_{\widetilde{A}}(\widetilde{Q})$. Thus it suffices to prove the analogue of (1) for $\widetilde{k}$, $\widetilde{\Delta}$ and $\widetilde{\sigma}$. We now define R, A, D' precisely. For sufficiently small r, there exists $R = (x_0, y_1) \in {}^cD$ such that $\operatorname{dist}(R, \Delta) \geqslant c_1$, and such that $\widetilde{\Delta}$ is the graph of a Lipschitz function with norm dominated by that of ϕ. This follows from the inverse function theorem, and the fact that the Jacobian (a_{ij}) of K near Δ can be written as

$$
a_{ij} = \begin{cases} c_1^{-2}\delta_{ij} + O(r), & i, j \leqslant n - 1 \\[2mm] -c_1^{-2}\delta_{ij} + O(r), & i = n \text{ or } j = n, \text{ where } x_n = y. \end{cases}
$$

Hence, we can choose $A = (x_0, y_0) \in D$ so that the analogue of (3) and (4) hold. Define $E = \{X | X = t(Q - A) + A, \text{ for } Q \in \Delta, t \geqslant 1; |X - A| \leqslant c_5\}$, $D' = {}^cE$. We can assume $D \subset D'$ and $\operatorname{dist}(R, \partial D') \geqslant c_0$. $K(D') \cup \{0\}$ is star-shaped with respect to $\widetilde{A}$ because the line through $\widetilde{A}$, $\widetilde{Q}$ corresponds under the (inverse) Kelvin transformation to the circle (or line) through A, Q, and R. Thus (5) is valid and we are reduced to the special case.

THEOREM 2. *If $f \in L^p(d\sigma)$, $p \geqslant 2$, then f has a harmonic extension u in D, with nontangential maximal function $N(u) \in L^p(d\sigma)$. Also, u converges nontangentially to f a.e. $d\sigma$.*

PROOF. The reduction in the proof of Theorem 1(b) shows that we may as well assume D is star-shaped. Let

$$
M(f)(Q) = \sup_{Q \in \Delta} \frac{1}{\omega(\Delta)} \int_\Delta f(\widetilde{Q}) \, d\omega(\widetilde{Q}).
$$

By [4, p. 315], $N(u)(Q) \leqslant c \, Mf(Q)$ a.e. $d\sigma$. The result now follows from Theorem 1(b) and the weighted norm inequalities of Muckenhoupt et al (see [1]).

FURTHER RESULTS. Let Ω be a star-shaped domain given by

$$
\Omega = \{(r, \theta) | 0 \leqslant r < \phi(\theta), \phi(\theta) \geqslant c > 0; \phi \text{ continuous}, \theta \in S^{(n-1)}\}.
$$

THEOREM 3. *Suppose that Ω is regular for the Dirichlet problem, and star-shaped as above.*

(a) *If $\nabla \phi \in L^p(d\theta)$, $p > 2$, then $\omega \ll \sigma$ and*

$$
k = d\omega/d\sigma \in L^q(d\sigma) \quad \text{for } q = 2(p - 1)/p.
$$

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