

ON THE UNION OF SETS OF SYNTHESIS AND DITKIN'S CONDITION IN REGULAR BANACH ALGEBRAS

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The questions have been raised, whether for every commutative semisimple regular Banach algebra with unit, the union of two sets of synthesis is again a set of synthesis, whether every set of synthesis is a Ditkin set, and whether a singleton which is of synthesis is a Ditkin set (cf. [4, pp. 31, 34, 41] and [2, p. 225]). The purpose of this note is to show that for the Banach algebra obtained by adjoining a unit to the Banach algebra introduced by Mirkil in [3], the answer to all three questions is in the negative. In fact Mirkil's proof of his main result [3, §5] contains implicitly a negative answer to the third (and therefore also to the second) question. Mirkil's algebra \mathbf{M} is defined as follows: Let \mathbf{T} denote the circle group identified with the interval $[-\pi, \pi)$, the group operation being addition modulo 2π . \mathbf{M} is the convolution algebra of all functions in $L^2(\mathbf{T})$ which are continuous on the interval $[-\pi/2, \pi/2]$ with norm defined by

$$\|f\| = \|f\|_2 + \sup\{|f(t)| : |t| \leq \pi/2\}.$$

(We recall that the convolution of two functions, f, g in $L^2(\mathbf{T})$ is defined by

$$f * g(t) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t - x)g(x) dx, \quad t \in \mathbf{T}.)$$

As shown in [3], \mathbf{M} is a Banach algebra in which the trigonometric polynomials are dense, whose regular maximal ideal space can be identified with the set of integers \mathbf{Z} , and its Gelfand representation is given by the Fourier transform $f \rightarrow \hat{f}$ where $\hat{f}(n) = (1/2\pi) \int_{\mathbf{T}} f(x)e^{-inx} dx, n \in \mathbf{Z}$.

We shall denote by \mathbf{M}_1 the Banach algebra obtained by the standard adjunction of unit to \mathbf{M} (\mathbf{M}_1 can also be regarded as the convolution algebra $\mathbf{M} \oplus \mathbf{D}$ where \mathbf{D} is the one dimensional vector space spanned by the unit point measure concentrated at τ). The maximal ideal space of \mathbf{M}_1 can be identified with $\mathbf{Z} \cup \{\infty\}$, the one point compactification of \mathbf{Z} .

We refer to [2, p. 230] for the definitions of sets of synthesis for regular Banach algebras (see also [4, p. 28] where these sets are called Wiener sets). We shall give here the definition directly for the algebras \mathbf{M} and \mathbf{M}_1 .

Let \mathbf{M}^* denote the dual space of \mathbf{M} . For every ν in \mathbf{M}^* we set $\hat{\nu}(n) =$

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$\langle e^{inx}, \nu \rangle$, $n \in \mathbf{Z}$, and define the spectrum of ν to be the set

$$\sigma(\nu) = \{n \in \mathbf{Z}: \hat{\nu}(n) \neq 0\}.$$

Every function g in $L^2(\mathbf{T})$ defines an element of \mathbf{M}^* by the mapping

$$f \longrightarrow \langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} f(x) \overline{g(x)} dx, \quad f \in \mathbf{M}.$$

We shall denote this element also by g .

DEFINITION 1. A set $E \subset \mathbf{Z}$ is called a set of synthesis for \mathbf{M} if every element in \mathbf{M}^* with spectrum contained in E , is in the w^* closure of the vector space spanned in \mathbf{M}^* by the set $\{e^{inx}: n \in E\}$. By the Hahn-Banach theorem, this is equivalent to the requirement that $\langle f, \nu \rangle = 0$ for every pair $\nu \in \mathbf{M}^*$ and $f \in \mathbf{M}$ such $\sigma(\nu) \subset E$ and $\hat{f} = 0$ on E .

DEFINITION 2. A closed set $K \subset \mathbf{Z} \cup \{\infty\}$ is called a set of synthesis for \mathbf{M}_1 if the set $K \cap \mathbf{Z}$ is of synthesis for \mathbf{M} .

Since the trigonometric polynomials are dense in \mathbf{M} , the empty set is of synthesis for \mathbf{M} , and therefore by Definition 2, the singleton $\{\infty\}$ is of synthesis for \mathbf{M}_1 . (It is easy to see that the usual definition of sets of synthesis for regular Banach algebras given in [2, p. 230] coincides, for the algebra \mathbf{M}_1 , with Definition 2.)

It is clear that the algebra \mathbf{M} is semisimple and regular and therefore the same is true for the algebra \mathbf{M}_1 .

We refer to [4, p. 30] for the definition of Ditkin sets (called there Wiener-Ditkin sets) for regular Banach algebras. We only note here that the condition for $\{\infty\}$ to be a Ditkin set for \mathbf{M}_1 , is equivalent to the requirement that for every function f in \mathbf{M} there exists a sequence of trigonometric polynomials $(p_n)_{n=1}^\infty$ such that $p_n * f \longrightarrow f$ in the norm of \mathbf{M} .

In what follows, we denote for every pair of integers r and s by $r\mathbf{Z} + s$ the set of all integers of the form $rn + s$, $n \in \mathbf{Z}$. The negative answers to the questions mentioned, is contained in the following.

THEOREM. (a) *The sets $K_1 = 4\mathbf{Z} \cup \{\infty\}$ and $K_2 = 4\mathbf{Z} + 2 \cup \{\infty\}$ are of synthesis for \mathbf{M}_1 but the set $K_1 \cup K_2$ is not of synthesis for \mathbf{M}_1 .*

(b) *The singleton $\{\infty\}$ is not a Ditkin set for \mathbf{M}_1 .*

PROOF OF (a). We have to show that the sets $E_1 = 4\mathbf{Z}$ and $E_2 = 4\mathbf{Z} + 2$ are of synthesis for \mathbf{M} but the set $2\mathbf{Z} = E_1 \cup E_2$ is not of synthesis for \mathbf{M} . For this we need first to identify the dual space \mathbf{M}^* . Let $C[-\pi/2, \pi/2]$ denote as usual the Banach space of complex continuous functions on $[-\pi, \pi/2]$ with the sup norm, and consider the Banach space $\mathbf{B} = L^2(\mathbf{T}) \times C[-\pi/2, \pi/2]$ with norm $\|(f, h)\| = \|f\|_2 + \|h\|_{C[-\pi/2, \pi/2]}$. Noticing that the mapping

$$f \longrightarrow (f, f|_{[-\pi/2, \pi/2]}), \quad f \in \mathbf{M}$$

(where $f|_{[-\pi/2, \pi/2]}$ denotes the restriction of f to $[-\pi/2, \pi/2]$) is an isometric isomorphism of \mathbf{M} onto a closed subspace of \mathbf{B} , using the Hahn Banach theorem, the fact that $\mathbf{B}^* = (L^2(\mathbf{T}))^* \times C^*[-\pi/2, \pi/2]$, and the known representations of the dual spaces of $L^2(\mathbf{T})$ and $C[-\pi/2, \pi/2]$, we see that every element ν in \mathbf{M}^* can be represented by a Borel measure on \mathbf{T} (which we also denote by ν) which admits a decomposition of the form $\nu = gdx + \mu$ where $g \in L^2(\mathbf{T})$ and μ is a Borel measure supported on $[-\pi/2, \pi/2]$, and the action of ν on \mathbf{M} is given by

$$\langle f, \nu \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} f(x) \overline{g(x)} dx + \int_{\mathbf{T}} f(x) d\bar{\mu}(x), \quad f \in \mathbf{M}.$$

Clearly $\|\nu\|_{\mathbf{M}^*} \leq \max\{\|g\|_2, |\mu|(\mathbf{T})\}$ where $|\mu|(\mathbf{T})$ is the total variation of μ . Suppose now that $\nu = gdx + \mu \in \mathbf{M}^*$ with g and μ as described above, and assume that $\sigma(\nu) \subset E_1 = 4\mathbf{Z}$. Then $\hat{\nu}(n) = \int_{\mathbf{T}} e^{-inx} d\nu(t) = 0$ for $n \notin 4\mathbf{Z}$ and therefore for every trigonometric polynomial p

$$\int_{\mathbf{T}} p(t + \frac{\pi}{2}) d\nu(t) = \int_{\mathbf{T}} p(t) d\nu(t),$$

and since a continuous function on \mathbf{T} is the uniform limit of trigonometric polynomials, the equality remains true if p is replaced by any such function. This shows that ν is a measure of period $\pi/2$, that is, $\nu(S + \pi/2) = \nu(S)$ for every Borel set $S \subset \mathbf{T}$. On the other hand, since μ is supported on $[-\pi/2, \pi/2]$, $\nu(S) = \int_S \overline{g(x)} dx$ for every Borel set $S \subset [0, \pi/2)$. Combining these facts we see that ν is in $L^2(\mathbf{T})$, and therefore setting $S_N(\nu) = \sum_{n=-N}^N \hat{\nu}(n) e^{inx}$, $N = 0, 1, \dots$, we deduce that

$$\|\nu - S_N(\nu)\|_{\mathbf{M}^*} \leq \|\nu - S_N(\nu)\|_2 \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This shows that every element in \mathbf{M}^* with spectrum contained in E_1 , is even in the norm closure in \mathbf{M}^* , of the vector space spanned in \mathbf{M}^* by the set $\{e^{inx} : n \in E_1\}$, and therefore E_1 is of synthesis for \mathbf{M} . To show that $E_2 = 4\mathbf{Z} + 2$ is also a set of synthesis for \mathbf{M} , consider a measure ρ in \mathbf{M}^* with spectrum in E_2 ; then the spectrum of the measure $e^{2ix}\rho$ is in E_1 , hence by the previous part of the proof this measure is in $L^2(\mathbf{T})$, and therefore ρ is also in $L^2(\mathbf{T})$. This implies as before, that E_2 is of synthesis for \mathbf{M} (even in the norm topology of \mathbf{M}^*). The remaining assertions of the theorem follow from the main result in [3]. For the sake of completeness we include here a different proof which is more in line with the approach of this paper. To show that the set $E = E_1 \cup E_2$ is not of synthesis for \mathbf{M} , consider the element of \mathbf{M}^* defined by the measure $\mu = \delta_{\pi/2} + \delta_{-\pi/2}$ (for every $\tau \in \mathbf{T}$, δ_τ denotes the unit point measure concentrated at τ) and the function f in \mathbf{M} defined by: $f(x) = 1$ for $|x| \leq \pi/2$ and $f(x) = -1$ for $\pi/2 < |x| \leq \pi$. Then $\hat{\mu}(n) = 2 \cos(n\pi/2)$, $\forall n \in \mathbf{Z}$, $\hat{f}(n) = 2 \sin(n\pi/2)/n\pi$, for $n \in \mathbf{Z} \setminus \{0\}$, and $\hat{f}(0) = 0$. Thus $\sigma(\mu) \subset E$ and $\hat{f} = 0$ on E , but $\langle f, \mu \rangle = 2$, and therefore E is not a set of synthesis for \mathbf{M} .

PROOF OF (b). To show that $\{\infty\}$ is not a Ditkin set for M_1 , consider again the measure μ and the function f defined above. Noticing that $\langle q, \mu \rangle = 0$ for every trigonometric polynomial q such that $\hat{q} = 0$ on $2\mathbb{Z}$, we obtain for every trigonometric polynomial p (by using the identity $p * f = \sum_{n=-N}^N \hat{p}(n) \hat{f}(n) e^{inx}$, where N is the degree of p) that

$$\|f - p * f\|_M \geq \|\mu\|_M^{-1} \langle f - p * f, \mu \rangle = 1$$

and consequently $\{\infty\}$ is not a Ditkin set for M_1 .

REMARK. The proof of the theorem shows that the sets K_1 and K_2 are of synthesis for M_1 even in the norm topology of M_1^* but $K_1 \cup K_2$ is not of synthesis for M_1 even in the w^* topology of M_1 .

The answers to the first and second questions mentioned at the beginning are not known for group algebras of locally compact noncompact abelian groups; in particular they are not known for the group algebra of \mathbb{Z} . A discussion of these problems and partial results can be found in [1, Chapter 1 and 2], [4, Chapters 2 and 6] and [5, Chapter 7].

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