BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 5, September 1979

TWO REDUCTIONS OF THE POINCARÉ CONJECTURE

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ABSTRACT. We study two reductions of the Poincaré conjecture. The first is group theoretic and is an improvement over Papakyriakopoulos' reduction [5]. The second reduces the conjecture to a special case of it.

We first examine Papakyriakopoulos' reduction and improve it. The method also gives a new proof of a crucial theorem in his reduction.

P. 1. CONJECTURE. Let G_p: $\{a_1, b_1, \ldots, a_p, b_p; \Pi_{i=1}^p [a_i, b_i], p > 1\}$ and let $Q_p = \{a_1, b_1, \ldots, a_p, b_p; \Pi_{i=1}^p [a_i, b_i], [a_1, b_1\tau]\}$, where $\tau \in [\Phi_p, \Phi_p]$, Φ_p being the free group generated by $a_1, b_1, \ldots, a_p, b_p$. Let T_p be an orientable surface of genus p and identity $\pi_1(T_p)$ with G_p. Then

(a) Q_p is torsion-free.

(b) The cover of T_p corresponding to the Kernel of the natural map φ_p: $G_p \longrightarrow Q_p$ is planar.

E. S. Rapaport proved [7] P.1.(a) and Papakyriakopoulos showed that P.1 implies the Poincaré conjecture [5]. He also considered the question ([5], [6]) whether P.1.(b) is group theoretic. Consider the following

P.2. CONJECTURE. The group Q_p defined above is a nontrivial free-product.

We will show

A. THEOREM. P.1 ⇒ P.2 ⇒ *Poincaré conjecture. Moreover,* P.1 *is group-theoretic.*

The crucial step in the reduction [5] of Poincaré conjecture to P.1 is a theorem which connects the problem of finding nontrivial simple loops in a certain normal subgroup with regular planar covers subordinate to it. This result was strengthened by Maskit in [4]; Lemmas 1 and 2 below imply his theorem. These lemmas connect the approach of Papakyriakopoulos with that of Stallings in [8].

Let

$$\{e\} \longrightarrow L \xrightarrow{i} G \xrightarrow{\varphi} H \longrightarrow \{e\}$$

be an exact sequence of groups, where G is the fundamental group of a closed

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Received by the editors March 20, 1979.

AMS (MOS) subject classifications (1970). Primary 57A10; Secondary 20E30, 20E40, 20J05.

Key words and phrases. Poincaré conjecture, planar cover, Heegaard decomposition, free product, group cohomology, structure theorem of Stallings.

0002-9904/79/0000-0406/$02.00

surface T. We are interested in finding a criterion which assests the existence of a simple loop on T which represents a nontrivial element of L.

1. LEMMA (a) *With the above notation there is a nontrivial simple loop on T whose nth power for some n represents an element of L if there is a factorization $\varphi = qp$: $G \xrightarrow{q} Q \xrightarrow{p} H$ such that $H^1(Q; Z_2Q) \neq 0$.*

(b) *If H is torsion-free, there is a nontrivial simple loop on T which represents an element of L if and only if φ factors through a nontrivial free-product.*

The proof of the above lemma is similar to the proof of Theorem 2 in [8]. For the first statement, we need also the structure Theorem from [9].

2. LEMMA. *With the notation of Lemma 1, there is a nontrivial simple loop on T whose nth power for some n represents an element of L, if there is normal subgroup K ($\neq \{e\}$) of G with $K \subset L$ and such that the cover of T corresponding to K is planar. The above condition is necessary* [4].

PROOF. Let Q be the quotient of G by K and let $\tilde{T}$ be the cover of T corresponding to K. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow H^1(Q; Z_2Q) & \longrightarrow & H^1(G; Z_2Q) & \longrightarrow & H^1(K; Z_2) & \longrightarrow \\ & & \downarrow & & \downarrow & \\ & & H^1_e(\tilde{T}; Z_2) & \longrightarrow & H^1(\tilde{T}; Z_2) & \\ & & \downarrow & & \downarrow & \\ & & H_1(\tilde{T}; Z_2) & \longrightarrow & H_1^\infty(\tilde{T}; Z_2) & \end{array}$$

The first row is exact (see [3, p. 354]). It is well known that the map in the second row is zero if and only if $\tilde{T}$ is planar. Since $K \neq \{e\}$, $H_1(\tilde{T}; Z_2) \neq 0$. Thus, if $\tilde{T}$ is planar we have $H^1(Q; Z_2Q) \approx H^1(G; Z_2Q) \neq 0$. This completes the proof using Lemma 1.

Next, let T be a two-sided orientable surface in a 3-manifold M. Let M_1, M_2 be the two parts into which T divides M; if $M - T$ is connected we write $M_2 = M_1$. We have two injections $i_1: T \longrightarrow M_1$ and $i_2: T \longrightarrow M_2$ and let φ_1, φ_2 be the induced maps in the fundamental groups. By Loop Theorem, $\varphi_i(T)$ is a free product of orientable surface groups and free cyclic groups and therefore $\varphi_i(T)$ is torsion-free. Consider $\varphi_1 \times \varphi_2: \pi_1(T) \longrightarrow \pi_1(M_1) \times \pi_1(M_2)$ and let φ denote the induced map of $\pi_1(T)$ onto the image H of $\varphi_1 \times \varphi_2$ and let L be the Kernel of φ. Then Lemma 2 implies the following.

3. THEOREM (PAPAKYRIAKOPOULOS, MASKIT). *With the above notation, there is a simple loop on T which represents a nontrivial element of L if and only if there is a normal subgroup $K \neq \{e\}$ of $\pi_1(T)$ with $K \subset L$ and such that the cover of T corresponding to K is planar.*

That P.1 $\Rightarrow$ Poincaré conjecture is proved using the above theorem when $(T; M_1, M_2)$ is a Heegaard decomposition of a homotopy 3-sphere M. In this case $\varphi_1 \times \varphi_2$ is surjective and the normal subgroup K_p generated by the image of $[a_1, b_1\tau]$ is in the Kernel of $\varphi_1 \times \varphi_2$ for some $\tau \in [\Phi_p, \Phi_p]$ (see [5]). Let φ_p, Q_p be as in P.1. Let $\widetilde{T}$ be the cover of T corresponding to K_p. As in Lemma 2, $\widetilde{T}$ is planar if and only if $H^1(Q_p; \mathbf{Z}_2Q_p) \longrightarrow H^1(G_p; \mathbf{Z}_2Q_p)$ is an isomorphism. This shows that the conjecture P.1 is group-theoretic. If P.1 is true, then

$$H^1(Q_p; \mathbf{Z}_2Q_p) \approx H^1(G_p; \mathbf{Z}_2Q_p) \approx H_1(K_p; \mathbf{Z}_2) \neq 0.$$

Since Q_p is torsion-free, by [9], Q_p is a free product. Thus P.1 ⇒ P.2. If Q_p is a free product, then by Lemma 1 we can find a simple loop on T which represents nontrivial element in $K_p \subset$ Kernel of $\varphi_1 \times \varphi_2$ and Poincaré conjecture follows by induction. This proves Theorem A.

The above proof shows that Papakyriakopoulos' approach (via Theorem 3 above) really involves factoring $\varphi_1 \times \varphi_2$ through a free product and hence it is contained in Stallings' approach [8]. However, Papakyriakopoulos has given possible candidates (K_p above) for finding simple loops and what we have done above seems to make his conjectures more accessible. The lemmas above can be generalized using a relative version of Stallings' theorem [10] and these can be used to prove Theorem 3 in Maskit's paper [4].

We next state another reduction of Poincaré conjecture. This reduction is based on the characteristic submanifold Theory ([1] and [2]) and the results of [11]. Thurston is reported to have strong results on the existence of hyperbolic structures for the class of manifolds $\mathcal{T}$ considered below and hence the new reduction may also be of some interest. Let $\mathcal{T}$ denote the following class.

$M \in \mathcal{T}$ if and only if M is compact, irreducible, $\chi(M) = 0$ and M does not admit essential embeddings of the annulus or torus.

C.1. CONJECTURE. If N is a simply connected 3-manifold obtained by adding solid tori to a $M \in \mathcal{T}$, then N is homeomorphic to S^3.

C.2. CONJECTURE. Similar to the above, restricting M to those which in addition satisfy

$\pi_1(M)$ is a subgroup of index $\leqslant 3$ in a real link group.

Our next reduction is

B. THEOREM. C.2 ⇒ Poincaré *conjecture.*

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