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THE HOMOTOPY TYPE OF HOMEOMORPHISMS OF 3-MANIFOLDS

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The purpose of this paper is to announce some results on homeomorphism groups of 3-manifolds. These results together with the Smale conjecture (a proof of which has recently been announced by A. Hatcher) essentially reduce the computation of the homotopy type of the homeomorphism group of a 3-manifold to an analysis of a certain associated configuration space (see §2) and the homotopy types of the homeomorphism groups of the prime factors. For a prime 3-manifold, this homotopy type is known (a) if the manifold is P^2-irreducible and sufficiently large [2], (b) for $S^1 \times S^2$ and $S^1 \underset{\sim}{\times} S^2$ (see §3).

The results announced here are an extension of the first part of the first author's Ph.D. thesis and use the methods of this thesis. None of our results depends on the Smale conjecture but we have indicated where the Smale conjecture simplifies the conclusion.

1. Preliminaries. Let M be a compact 3-manifold (possibly with boundary) and let $\mathcal{H}(M)$ denote the group of PL homeomorphisms of M fixed on ∂M. This group has the same homotopy type as the (topological) homeomorphism group of M (fixed on ∂M) and also, assuming the Smale conjecture, as the diffeomorphism group of M.

Let $\mathcal{H}(M, D) \subset \mathcal{H}(M)$ denote the subgroup of homeomorphisms which are in addition fixed on a standard 3-disc D in int M. We remind you that there is a fibration:

$$\mathcal{H}(M, D) \subset \mathcal{H}(M) \longrightarrow \mathcal{E}(D, M)$$

where $\mathcal{E}(D, M)$ denotes the space of PL embeddings of D in int M. $\mathcal{E}(D, M)$ has the homotopy type

$$M \times PL_3 \quad \text{if } M \text{ is orientable,}$$

$$M \underset{\sim}{\times} PL_3 \quad \text{if } M \text{ is nonorientable,}$$

where the twisted product is defined by interchanging the components of PL_3 around orientation reversing loops in M.

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We also remind you that $PL_3 = H(\mathbf{R}^3, 0)$ and that the Smale conjecture is equivalent to the homotopy equivalence $O_3 \simeq PL_3$.

The automorphism groups of M and (M, D) are defined by

$$\mathrm{Aut}\, M = \pi_0(H(M)), \qquad \mathrm{Aut}(M, D) = \pi_0(H(M, D)).$$

2. The main theorem. Let

$$M = P_1 \# \cdots \# P_n \#_j S^1 \underset{(\sim)}{\times} S^2$$

be a decomposition of M into prime factors, where the P_i are irreducible and $\#_j S^1 \times_{(\sim)} S^2$ denotes the connected sum of j copies of $S^1 \times S^2$ or the twisted S^2-bundle $S^1 \underset{\sim}{\times} S^2$ over S^1.

THEOREM 1. *There is a homotopy equivalence*

$$H(M, D) \simeq \mathop{\times}_{i=1}^{n} H(P_i, D) \times (\mathop{\times}_{j} \Omega PL_3) \times \Omega C.$$

Here $\times_j \Omega PL_3$ denotes the product of j copies of the loop space on PL_3 and ΩC is the loop space on an associated configuration space C defined roughly as follows: a point of C corresponds to a presentation of M as a connected sum or to a presentation of a 3-manifold with the same prime factors as M. More precisely, each factor of M is supposed parallelised (or the orientation double cover is parallelised for the non-orientable factors); this allows us to speak of standard discs of either orientation at any point of M. We start with $S^3 = D^3_- \cup D^3_+$ and choose a number of points $p_1, \ldots, p_t$, $t \leqslant n$, and unordered pairs of points $q_1, r_1, \ldots, q_s, r_s$, $s \leqslant j$, all distinct lying in int D^3_+. Let $D = D^3_-$. At each point standard disjoint discs are chosen and a selection of prime factors $P_{i_1}, \ldots, P_{i_t}$ are glued in at $p_1, \ldots, p_t$ using the chosen discs. The discs corresponding to q_i, r_i are cut out and the boundaries glued together to form s handles, which may be twisted. In the resulting manifold more points $p_{t+1}, \ldots, p_u$, $u \leqslant n$, and pairs $q_{s+1}, r_{s+1}, \ldots, q_v, r_v$, $v \leqslant j$, are chosen and the process is repeated using factors of M not used at the first stage. The process stops when all the factors of M have been used. This set of choices is a point of C and identifications are made corresponding to making the choices in a different order and to the possibility of two or more of the P_i being the same and so on. A point of C thus gives a 3-manifold which may be homeomorphic to M (i.e. if all the orientations were correct in the case when M is orientable). The identifications can be summarised by saying that two points are identified if the resulting connected sum decompositions are the same (in other words the same or isomorphic factors are glued in the same places). The base point of C is determined by the given decomposition of M.

Information on $\pi_1(C) = \pi_0(\Omega C)$ will be given in §5.

3. The case $M = S^1 \times_{(\sim)} S^2$. In this case C has the homotopy type of the real projective plane P^2 and we deduce:

COROLLARY 1.

$$H(S^1 \underset{(\sim)}{\times} S^2, D) \simeq \Omega P^2 \times \Omega PL_3.$$

We can also determine the homotopy type of $H(S^1 \times_{(\sim)} S^2)$:

THEOREM 2. (a) *$H(S^1 \times S^2) \simeq H(S^1) \times Q$ where $Q \subset H(S^1 \times S^2)$ is the space of homeomorphisms that send the base point to the 2-sphere over the base point and respect the orientation on S^1, and there is a fibration*

$$\Omega PL_3 \longrightarrow Q \longrightarrow PL_3$$

which splits if the Smale conjecture is true.

(b) *There are fibrations:*

$$\widetilde{Q} \longrightarrow (S^1 \underset{\sim}{\times} S^2) \longrightarrow H(S^1), \quad \Omega PL_3 \longrightarrow \widetilde{Q} \longrightarrow PL_3$$

where $\widetilde{Q} \subset H(S^1 \underset{\sim}{\times} S^2)$ is the subspace of the homeomorphisms which send the base point to the 2-sphere over the base point and respect the orientation on S^1.

4. The case where the factors P_i are P^2-irreducible sufficiently large. Using Hatcher [2] and the fibration in §1 we can deduce that

$$\pi_n H(P_i, D) \cong \pi_{n+1} PL_3, \qquad n \geqslant 1,$$

and that

$$0 \longrightarrow Z_2 \longrightarrow \pi_0 H(P_i, D) \longrightarrow \mathrm{Aut}^+ \pi_1 P_i \longrightarrow 0$$

is exact, where $\mathrm{Aut}^+ \pi_1 M$ is the group of automorphisms which respect the orientation homomorphism. For P_i orientable $\mathrm{Aut}^+ \pi_1 P_i \cong \pi_0 SH(P_i, x_0)$ [2] where $SH(P_i, x_0)$ is the subgroup of the orientation preserving homeomorphisms fixed on a point x_0 in int P_i.

COROLLARY 2. *If $M = \#_j S^1 \times_{(\sim)} S^2 \#_i P_i$ where the P_i are P^2-irreducible sufficiently large then*

$$\pi_n H(M, D) \cong \pi_{n+1} C \oplus \Big(\bigoplus_{i+j} \pi_{n+1} PL_3\Big), \qquad n \geqslant 1.$$

5. Application to the automorphism group. We now describe certain specific automorphisms of M.

(1) PERMUTATIONS. Choose two factors P_i and P_j which are homeomorphic. Slide these factors inside D^3_+ without meeting the other factors to interchange them. This corresponds to a loop in C which interchanges the corresponding points p_i, p_j. There are similar "permutations" using two handles or two ends of the same handle.

(2) SLIDES. Choose a factor P_i, say, and a loop in M with P_i and D removed. Slide P_i around this loop. This corresponds to the loop in C in which

p_i is taken around the chosen loop. There are similar "slides" using one end of a handle.

THEOREM 3. *The image of* $\pi_0(\Omega C)$ *in* $\mathrm{Aut}(M, D)$ *is generated by slides and permutations and hence every automorphism (orientation preserving if M is orientable) of a 3-manifold is a composition of slides, permutations and automorphisms of the factors.*

Now consider the homeomorphism $\mathrm{Aut}(M, D) \xrightarrow{\pi} \mathrm{Aut}\, \pi_1 M$ that associates to a homeomorphism the induced automorphism on π_1. It is possible to describe explicitly the automorphisms of $\pi_1 M$ corresponding to slides and permutations. In fact, $\mathrm{Aut}\, \pi_1 M$ is generated by the images of slides, permutations and by the automorphisms of the fundamental groups of the factors and it follows from Theorem 3 and [1] that

COROLLARY 3. *There is an injection*

$$\pi_0(\Omega C) \longrightarrow \mathrm{Aut}\, \pi_1 M.$$

Combining Corollary 3 with Waldhausen's results (see [2]) we can deduce:

COROLLARY 4. *If* $M = \#_i P_i \#_j S^1 \times_{(\sim)} S^2$ *where the* P_i *are* P^2*-irreducible sufficiently large, then there is an exact sequence*

$$0 \longrightarrow \bigoplus_{i+j} Z_2 \longrightarrow \mathrm{Aut}(M, D) \xrightarrow{\pi} G \longrightarrow 0$$

where G is a subgroup of $\mathrm{Aut}\, \pi_1 M$ *which can be described explicitly.*

Each $\mathbf{Z}_2$-factor in the kernel of π comes either from a rotation parallel to the separating sphere of an irreducible factor or from a rotation parallel to the belt-sphere of a handle (see also [3] and [4]).

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