REFERENCES

1. J. Horváth, *Topological vector spaces and distributions*, Addison-Wesley, Reading, Mass., 1966.
2. A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge Tracts in Mathematics and Physics, no. 53 Cambridge Univ. Press, New York, 1964.
3. H. H. Schaefer, *Topological vector spaces*, MacMillan, New York, 1966.
4. F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

L. WAELBROECK

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 6, November 1978

Transcendental number theory, by Alan Baker, Cambridge Univ. Press, New York, 1975, x + 147 pp., $13.95.

Lectures on transcendental numbers, by Kurt Mahler, Edited and completed by B. Diviš and W. J. LeVeque, Lecture Notes in Math., no. 546, Springer-Verlag, Berlin, Heidelberg, New York, 1976, xxi + 254 pp., $10.20.

Nombres transcendants, by Michel Waldschmidt, Lecture Notes in Math., no. 402, Springer-Verlag, Berlin and New York, 1974, viii + 277 pp., $10.30.

The last dozen years have been a golden age for transcendental number theory. It has scored successes on its own ground, while its methods have triumphed over problems in classical number theory involving exponential sums, class numbers, and Diophantine equations. Few topics in mathematics have such general appeal within the discipline as transcendency. Many of us learned of the circle squaring problem before college, and became acquainted with Cantor's existence proof, Liouville's construction, and even Hermite's proof of the transcendence of e well before the close of our undergraduate life. How can we learn more?

Sophisticated readers may profitably consult the excellent survey articles of N. I. Feldman and A. B. Shidlovskiĭ [9], S. Lang [12], and W. M. Schmidt [17]. I will begin by addressing the beginner who has a solid understanding of complex variables, basic modern algebra, and the bare rudiments of algebraic number theory (the little book of H. Diamond and H. Pollard [8] is more than enough). My first advice is to read the short book of I. Niven [14] for a relaxed overview of the subject. If the reader is impatient, he may take Chapter 1 of Baker for an introduction. Either way he will learn short proofs of the Lindemann-Weierstrass theorem, that if the algebraic numbers $\alpha_1, \ldots, \alpha_n$ are distinct, then

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0$$

for any nonzero algebraic numbers $\beta_1, \ldots, \beta_n$. As special cases of this e and π are transcendental. These proofs are unmotivated; Baker mentions that they stem from the problem of approximating e^x by rational functions of x, and refers the reader to Hermite's original papers. At this point the reader may also find it most enjoyable and enlightening to turn to the appendix of Mahler's book where a thorough discussion of most of the classical proofs for

e and π is given. But our age is not history conscious, and the reader will probably ask "what comes next?"

As a *function* of z, the exponential $e^z = \Sigma z^n/n!$ is transcendental, and there is only one algebraic point at which it assumes an algebraic value, namely $z = 0$. It seems natural to conjecture that any entire transcendental function $f(z) = \Sigma a_n z^n$ with the a_n rational is transcendental at all but finitely many algebraic points. However, a sparkling counter-example by P. Stäckel [19] (based on a clever use of Cantor's diagonal enumeration procedure) shows there is such a function such that it and all of its derivatives are algebraic at every algebraic point! This is discussed in Chapter 3 of Mahler, where various more elaborate counter-examples and some related open questions are investigated. I think Stäckel's example belongs in Baker's treatise as well (it would fit into the blank space at the bottom of p. 8). It is in fact a common phenomenon that the transcendence theory of functions is easier than that of function values (witness the status of the Littlewood and Schanuel conjectures (Baker, pp. 104, 120)). On the other hand, the Stäckel function is very far removed from the functions that arise naturally in mathematics and physics, so one is tempted to modify the old conjecture by putting growth conditions on the numerators and denominators of the a_n, and requiring that $f(z)$ satisfy a functional equation or a linear differential equation with "nice" coefficients. The latter leads us into the Siegel-Shidlovskiĭ theory of E-functions.

An E-function is a series $\Sigma a_n z^n/n!$ such that (i) $a_0, a_1, \ldots$ are elements of an algebraic number field, and (ii) a sequence of positive integers $b_0, b_1, \ldots$ exists such that $b_n a_0, b_n a_1, \ldots, b_n a_n$ and b_n are all algebraic integers whose conjugates are at most $C(\varepsilon)n^{\varepsilon n}$ for any $\varepsilon > 0$. We restrict our attention to E-functions that satisfy a system of homogeneous linear differential equations

$$y_i' = \sum_{j=1}^{n} f_{ij}(x) y_j, \qquad 1 < i < n$$

where the coefficients of all the E's and f's belong to an algebraic number field K. What makes them so tractable is the fact that sums and products of E-functions are again E-functions satisfying similar systems of differential equations.

THEOREM. *If the functions $E_1(x), \ldots, E_n(x)$ are algebraically independent over $K(x)$, then for all but finitely many algebraic numbers α, the values $E_1(\alpha), \ldots, E_n(\alpha)$ are algebraically independent.*

In fact, the exceptions are at worst zero and the poles of the f_{ij}.

Mahler's proof of the above theorem takes about 82 pp., while Baker uses only 6 pp.! The urge to turn to Baker is irresistable, but his dazzling extraction of information from determinantal equations may overwhelm the reader. Throughout his book Baker is elegant, concise, precise, quite complete, and the devices of his proofs are even more elementary than anyone else's. His every word is chosen as carefully as a word in an A. E. Housman poem. But the lack of redundancy and the absence of occasional stretches of trivia can wear down the reader. I add that Baker does somewhat prepare the reader for Siegel-Shidlovskiĭ in the short previous chapter (of independent interest) where he proves his outstanding result on simultaneous approximation to $e^{\theta_1}, \ldots, e^{\theta_n}$ (where the θ_i are nonzero rationals).

Mahler has analyzed rather than synthesized the Siegel-Shidlovskiĭ proof. He carefully identifies many relevant vector spaces, and does all the differential algebra in a very systematic fashion, putting one little lemma after another until the nonvanishing of Shidlovskiĭ's determinant is proved. This determinant must be "not too small" since it is an algebraic number, but it is quite small by the nature of its construction. By comparing upper and lower estimates, one shows that a certain space spanned by function values has dimension "not much smaller" than a certain space spanned by functions. The theorem is proved by applying this fact to a space spanned by products of the original E-functions (a weak version of Hilbert's theorem on forms is also needed; Mahler proves it *ab initio*). Afterwards Mahler puts in an additional 37 pp. on applications of the Siegel-Shidlovskiĭ theory (this is almost missing in Baker). Not only does one learn many "little" facts of interest from Mahler that may be of use elsewhere, but one can also identify subproblems of research interest (see Mahler, p. 113). (Additional remarks: most of pp. 151–153 in Mahler correspond to the word "Plainly" on p. 11 of Baker; much of p. 97 in Mahler corresponds to the phrase "readily verified" on p. 110 of Baker.) Mahler and Baker both agree that the main outstanding problem here is to generalize the theory to a function class wider than E-functions.

In going from Lindemann-Weierstrass to Siegel-Shidlovskiĭ the reader should be warned that a somewhat artificial (though ingenious and by now familiar) element has entered. The Siegel-Shidlovskiĭ proof requires that a certain linear combination of E-functions (with polynomials over the integers as coefficients) vanishes at 0 to a high order. In the work of the Lindemann-Weierstrass era for the E-functions $e^{\alpha_i z}$, with α_i algebraic, these polynomials were determined in an explicit fashion, and reflected intrinsic properties of the functions. In the work of Siegel (and also in earlier work by Thue on a kindred subject) one simply expresses the desideratum as a system of a limited number of linear equations with a large number of undetermined coefficients and shows (via the pigeonhole principle) that with enough coefficients a solution in integers *exists*. There is nothing nonconstructive here, but something is lost. It would be nice if someone could at least do the algebraic independence of values of the Bessel functions without this device, especially since it often gives much stronger results on how well a transcendental number can be approximated by algebraic numbers. One very readable presentation of an "intrinsic method" is in Bundschuh [**4**]; other examples of this technique (that applies also to approximation of algebraic irrationals by rationals) are given in Mahler [**13**] and Baker [**2**]. Baker's proof that $2^{1/3}$ cannot be too well approximated is especially noteworthy, though mentioned only briefly in his book. The rest of the subject "suffers" from more and more ad hoc ingenuity.

The reader may next be curious about Hilbert's seventh problem, to demonstrate that aside from trivial cases the number α^β is transcendental if α and β are both algebraic and β is not a real rational. For example, $2^{\sqrt{2}}$ ought to be transcendental. This problem was solved affirmatively in the mid 1930s by A. O. Gelfond and (independently) T. Schneider. The result has now been brilliantly extended by Baker in several useful ways, both qualitatively and quantitatively. Qualitatively we have (Baker, p. 10)

THEOREM. *If* $\alpha_1, \ldots, \alpha_n$ *are nonzero algebraic numbers such that* $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over the rationals, then* 1, $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over the field of all algebraic numbers.*

A host of previously unsolved transcendence questions can now be handled with ease. For example, one can show that $\pi + \ln 3$ is transcendental. Now let's examine the proof. The reader can easily go to Lemma 2 on p. 14 of Baker, but may then be jolted by the intricate ad hoc auxiliary function Φ. And this is only the start of things to come. There is no superfluous chatter in Baker–his remark on p. 13 that "the inclusion of 1 in Theorem 2.1 ... entails a relatively large amount of additional complexity ... " should be taken to heart. The immediate remedy is to turn to Baker's paper [1] where the simpler (but still quite profound) version without the 1 is proved. But there is another path to understanding.

Waldschmidt's book has not been mentioned yet. It omits the oft told tale of Lindemann-Weierstrass, except to state that result on p. 206, and concentrates almost entirely on applications of modern ad hoc methods to exponential functions alone. This book is addressed to the beginner, and starts with an excellent survey of prerequisite techniques in Chapter 1. Two solutions of Hilbert's 7th problem are given; that of Schneider in Chapter 2 and that of Gelfond in Chapter 3. Perhaps the place to start is in Chapter 3 on p. 71, where the Hermite-Lindemann theorem (α and e^α are not both algebraic, unless $\alpha = 0$) is proved as a warm-up. Assume α and e^α are both algebraic. Define $F = F_N(z)$ by $F = P(x,y)$ and $x = z$, $y = e^z$, where $P(x,y)$ is a polynomial in x and y of degrees $d_1(N)$ and $d_2(N)$ respectively. Assume that N is large, and that both $d_1(n)$ and $d_2(n)$ tend to ∞ as $n \to \infty$. By the pigeonhole principle we can find coefficients for P that are not very large such that the first N derivatives of F vanish at $z = 0$ and $z = \alpha$. For suitable large M the ratio

$$G_N(z) = F_N(z)/\left[z^M(z-\alpha)^M\right]$$

is entire, and doesn't vanish for at least one of α and e^α (say α). Now use the technique of Schwarz's lemma (i.e. the maximum modulus principle) to show $G_N(z)$ is small when $|z| \leqslant R_M$, where R_M is a suitable function of M that tends to ∞ as $M \to \infty$. Hence $|G_N(\alpha)|$ is small. On the other hand, a nonzero algebraic number cannot be too small by a simple extension of the fact that there is no integer between 0 and 1. Thus we have both upper and lower bounds on $|G_N(\alpha)|$. A contradiction is obtained by suitable choice of parameters. (Incidently, the motivation for choice of parameters is done very well in Waldschmidt, and there is also an excellent discussion of what further results can be squeezed out of this method.)

The reader now knows a fundamental technique: use a (false) assumption that certain function values are algebraic and the Thue-Siegel pigeonholing technique to create a polynomial in the functions that has large numbers of zeros (or zeros of high multiplicity, or both). Use the zeros à la Schwarz to show that the function is rather small, and oppose this to the fact that an algebraic number of fixed height cannot be too small.

Even now **[1]** is tough. The reader may next wish to study pp. 102–109 in Gelfond **[10]**, where a development of this idea is presented that might be called the "method of contagious zeros". Instead of finding directly a point at which the appropriate auxiliary function does not vanish (at least not to too great a multiplicity) we look at certain points near those at which it vanishes, and at which its value is algebraic. Since it must be quite small here, it must be 0 at these additional points. Now we have greatly increased the number of zeros, and may either apply our previous method (from a much improved vantage point) or iterate the current procedure again! Now the reader is well prepared to appreciate the motivation of Baker's method (which uses a large number of iterations) given on pp. 229–234 of Waldschmidt. The simple diagram on p. 233 (due to Baker) is quite helpful; I would have been happy to see it also in Baker's book. Upon reading **[1]** the reader should have confidence that he can absorb the proof (Baker, pp. 14–21; Waldschmidt pp. 234–238) with full understanding.

In Chapter 3 Baker goes on to prove in detail quantitative versions, in which explicit lower bounds are given for the (now known not to vanish) linear forms in logarithms. This leads to his spectacular applications to Diophantine equations and class numbers in Chapters 4 and 5. In Chapter 6 we see that his methods also apply to elliptic functions.

Before investigating further topics, I have two comments on notation and style. One traditionally measures the "size" of an algebraic number α by means of its height $H(\alpha)$ and degree $d(\alpha)$. The height is the maximum of the absolute values of the coefficients of the minimal polynomial of α over the integers. Thus, a fundamental problem of Diophantine approximation is how small $|x - \alpha|$ can be in terms of H and d, given that $H(\alpha) \leq H$ and $d(\alpha) \leq d$. In **[11]**, S. Lang defines

$$s(\alpha) = \text{size } \alpha = \max(\log b, |\sigma\alpha|)$$

where b is the smallest positive rational integer such that $b\alpha$ is integral, and $\sigma\alpha$ runs over all conjugates of α. For an α of fixed degree, a bound on $H(\alpha)$ yields a bound on $s(\alpha)$ and vice versa. Since (p. 6, Waldschmidt)

$$-2d(\alpha)s(\alpha) \leq \log|\alpha|,$$

a bound on $s(\alpha)$ implies that $|\alpha|$ cannot be too small. Waldschmidt follows Lang; we shall refer to their size by the French word *taille*. They also introduce $t(P)$, the taille of the polynomial P, by

$$t(P) = \max(\log H(P), 1 + \deg P)$$

(p. 27, Waldschmidt). The taille function satisfies various convenient inequalities, and its use leads to somewhat slicker proofs. I feel that height has a much more direct intuitive appeal, and that introducing taille is a pedagogical error. Baker and others get along quite well without it. However, this is a matter of taste, and the concept is certainly most appropriate in Chapter 5 of

Lang and Chapter 4 of Waldschmidt. Also, Waldschmidt does mention the alternatives in Exercise 4, p. 28. My second comment is that Waldschmidt sometimes does too good a job of estimating constants where a much weaker estimate, or even a 0 or $\ll$ symbol would suffice; in contrast, Lang goes to the other extreme. This again is a matter of taste. Something can be said for writing everything out explicitly. For example, although I don't see that pp. 150–152 of Waldschmidt give anything more than Baker's instant resolution of the problem on p. 124, Lemma 4, I do feel that readers who breeze through Waldschmidt's Lemma 5.3.1 on pp. 147–149 might have gotten stuck on Lemma 3, p. 123 of Baker (which is the same).

The question of whether a real number x is transcendental is a special case of the question "how well can x be approximated by algebraic numbers in terms of height and degree?" Obviously, one begins by studying how closely a fixed algebraic number can be approximated by algebraic numbers of lower degree. The theorem of Liouville is a first step, and its contrapositive yields explicit transcendental numbers. Liouville's work was successively deepened by Thue, Siegel, Dyson, Gelfond, Schneider, Roth and Wirsing. At present the high point of the theory is the following consequence of a deep n-dimensional theorem of W. M. Schmidt.

THEOREM. *If $\varepsilon > 0$ and $d(\alpha) \geqslant n + 1$, where α is algebraic, then there are only finitely many algebraic numbers β with $d(\beta) \leqslant n$ such that*

$$|\alpha - \beta| < H(\beta)^{-n-1-\varepsilon}.$$

Aside from the ε, the exponent is known to be best possible as a function of n. Waldschmidt omits this topic. Mahler handles only the Liouville level (Chapter 1) but does it superbly. Baker jumps directly into the proof of Schmidt's theorem, and finishes it completely in 15 1/2 pp., a miracle of exposition and brevity. (When the manuscript of Schmidt's proof first became available, it provided a Diophantine approximation seminar at the University of Illinois with material for an entire semester. Our impression was that it was very tightly organized, without a trace of fat.) I do feel that beginners are best advised to first master Thue's theorem (see Davenport [7] for an excellent exposition) and then Roth's theorem (see, for example, [5] or [16]).

One may now attempt to classify all real numbers according to how well they can be approximated by algebraic numbers. The classification of Mahler (see Chapter 8 of Baker) into A, S, T, and U numbers (each class of which has been further subdivided by later investigators, especially Mahler himself) seems the most illuminating. Here the spectrum extends from the hardest to approximate numbers at the A (or algebraic) end, to the easiest at the U (or Liouville) end. It is not at all obvious that there are no gaps in the spectrum. In fact, the mere existence of T numbers was not established until Schmidt proved this via Wirsing's theorem (Schmidt's own stronger theorem still seems to leave open the possibility of a mini-gap in the T-spectrum: see Baker, pp. 92–94). It is not hard to show that almost all real numbers are S numbers. The main result here is Sprindzuk's theorem that almost all are S numbers of type 1. This is the theme of Sprindzuk's book [18]; here Baker does somewhat more in only eight pages!

Some very striking results on the algebraic independence of certain constants were obtained independently by Brownawell and Waldschmidt in the early 1970s. In particular, they showed that at least one of the numbers $\exp(e)$, $\exp(e^2)$ is transcendental. A thorough and extensive account of this work (partly based on Tijdeman's estimate of the number of zeros of an exponential polynomial) is found in Chapters 6 and 7 of Waldschmidt. For a briefer account, see Chapter 12 of Baker.

I am now ready to size up these books. Mahler's is primarily an attempt to expose the Siegel-Shidlovskiĭ theory in complete detail, in an appropriate framework, so that it is accessible to the general mathematical public. He succeeds admirably. In fact, the disclaimer on the back cover, to the effect that these are only rough notes, is quite inappropriate. Few books are written with greater clarity and precision. I noticed only very minor misprints and obscurities.

On p. xii, line 15, the bottom letter of the right hand binomial coefficient should be D, not P; on p. 102, line 16, "wity" should be "with"; on p. 133, line 17, the subscript on the last D should be upper case; on p. 134, second line from the bottom, the assumption should be $P(f_1(\alpha), \ldots, f_m(\alpha)) = 0$. On p. 146, line 7, one should assume $n \geqslant p^t$; also, line 10 should begin $\leqslant |a| + (n-1)b$. On p. 184, line 14, the first reference should be to Mahler (1968a); on p. 166 lines 11 and 12 are incoherent. There is no index, but the table of contents is quite detailed.

Mahler's book also points out possibilities for further research, especially by giving some partial results of J. Popken on functions satisfying *algebraic* rather than linear differential equations.

Waldschmidt's book is the only one of the three obviously designed as a textbook, with a gentle pace, lots of exercises of varying degrees of difficulty, clear motivation, and an index. There are also suggestions for future research. By design, Waldschmidt focuses almost exclusively on the exponential function. Instructors basing a course on this text may wish to supplement it with, for example, the theorems of Liouville, Thue-Siegel-Roth, and Lindemann-Weierstrass. Although my ability to proofread a French text is rather limited, the favorable remarks above concerning Mahler's book seem to apply here as well. This is an excellent and practical introduction to modern transcendental number theory.

Baker's book is *the* book on transcendental numbers. In only 128 pp. he covers a majority of those areas that have reached definitive results, presents most of the proofs in a *complete* yet far more compact form than hitherto available, and covers historical and bibliographical matters with great thoroughness and impeccable scholarship. As literature, it compares well with the finest works of Landau, Rademacher, and Titchmarsh. From its encyclopedic coverage of the field I would have guessed it to be 400 pages at least; the actual small size also means that it is not expensive. Moreover, it is beautifully printed with large type, and seems devoid of misprints. (Some early copies had an obvious misprint on p. 85. Pages 141–144 are simply *absent* in the review copy–I trust this was an isolated incident.) One thing I did miss in Baker's book was the sort of freewheeling speculation provided by Lang in **[11]**.

Despite the Herculean labors it has inspired, transcendence theory is still in its infancy. The mere irrationality of Euler's constant, the number $e + \pi$, and even $\zeta(3)$ remain undecided. Perhaps we are presently like pre-Newtonian mathematicians, who with their method of exhaustion could determine certain areas, but never really grasped the fundamental theorem.

Room remains for more books at all levels. Among the great difficulties in developing and organizing this subject is to find a really appropriate level of generality. I get the impression that Lang feels it is fairly high. However, most of the strong results achieved hitherto have been accomplished with suprisingly elementary (though not easy) methods. To emphasize our ignorance again, I point out an interesting countable field K investigated by Corzatt **[6]** that is much larger than the field of all algebraic numbers, and is *not* defined simply by "throwing a set of numbers into $\mathbf{Q}$ and taking algebraic closure." Here no one seems to be able to construct a finite set of numbers such that some one of them is not in K; it might even be nontrivial if finite is replaced by countable. There is also room for further development of the various fundamental "lemmas about polynomials" that the subject relies on (see Chapter 1 of either Mahler or Waldschmidt; Baker disperses them throughout his book). A significant development here is the theorem of Per Enflo on heights of products of polynomials in several variables. This work has been further developed and also simplified by H. L. Montgomery.

In another direction, no one seems to have inquired into the arithmetical nature of the radii of convergence of the power series that arise in Pólya's theory of counting [3], [**15**].

I end by quoting a conjecture whose importance in transcendental number theory is comparable to that of the Riemann hypothesis in analytic number theory (Baker, p. 120; Waldschmidt, p. 208).

CONJECTURE (SCHANUEL). If the complex numbers $x_1, \ldots, x_n$ are linearly independent over the rationals $\mathbf{Q}$, then the field

$$\mathbf{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

has transcendence degree at least n.

REFERENCES

1. A. Baker, *Linear forms in the logarithms of algebraic numbers*. I, II, Mathematika **13** (1966), 204–216; **14** (1967), 102–107.

2. ______, *Rational approximations to $2^{1/3}$ and other algebraic numbers*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 375–383.

3. E. A. Bender, *Asymptotic methods in enumeration*, SIAM Rev. **16** (1974), 485–515.

4. P. Bundschuh, *Irrationalitätsmasse für e^a, $a \neq 0$ rational oder Liouville-Zahl*, Math. Ann. **192** (1971), 229–242.

5. J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Univ. Press, Cambridge, London and New York, 1965.

6. C. E. Corzatt, *A large algebraically closed field*, Proc. Amer. Math. Soc. **46** (1974), 191–194.

7. H. Davenport, *A note on Thue's theorem*, Mathematika **15** (1968), 76–87.

8. H. Diamond and H. Pollard, *The theory of algebraic numbers*, 2nd ed., Carus Math. Monograph, no. 9, Mathematical Association of America, Wiley, New York, 1975.

9. N. I. Feldman and A. B. Shidlovskiĭ, *The development and the present state of the theory of transcendental numbers*, Russian Math. Surveys **22** (1967), 1–79.

10. A. O. Gelfond, *Transcendental and algebraic numbers*, Dover, New York, 1960.

11. S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, Reading, Mass., 1966.

12. ______, *Transcendental numbers and diophantine approximations*, Bull. Amer. Math. Soc. **77** (1971), 635–677.

13. K. Mahler, *Ein Beweis des Thue-Siegelschen Satzes über die Approximation algebraischer Zahlen für binomische Gleichungen*, Math. Ann. **105** (1931), 267–276.

14. I. Niven, *Irrational numbers*, Carus Math. Monograph, no. 11, Mathematical Association of America, Wiley, New York, 1956.

15. G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen and chemische Verbindungen*, Acta Math. **68** (1937), 145–254.

16. K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20; Corrigendum, ibid., 168.

17. W. M. Schmidt, *Approximation to algebraic numbers*, Enseignement Math. **17** (1971), 188–253.

18. V. G. Sprindzuk, *Mahler's problem in metric number theory*, Transl. Math. Monographs, no. 25, Amer. Math. Soc., Providence, R. I., 1969.

19. P. Stäckel, *Ueber arithmetische Eigenschaften analytischer Funktionen*, Math. Ann. **46** (1895), 513–520.

KENNETH B. STOLARSKY

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 6, November 1978

First order categorical logic, by Michael Makkai and Gonzalo E. Reyes, Lecture Notes in Math., vol. 611, Springer-Verlag, Berlin, Heidelberg, New York, 1977, viii + 301 pp. $14.30.

The authors obtained new proofs for theorems of Barr and Deligne concerning topoi, and also obtained some basic new results in this area. Their proofs are applications of standard results and techniques of logic. Since the relationships between logic and category theory at this relatively deep level are not widely known, they wisely decided to give a general and self-contained explanation of these relationships in addition to their proofs and results. The resulting work, despite its unfinished character typical of the lecture notes series, should hence be, at least in part, of considerably wider interest than a research paper on these results would have been.

Before describing briefly the contents of the book, it is desirable to make some remarks on the role of categorical logic in mathematics. Categorical logic may be described as one of the algebraic ways of looking at logic. Algebraic logic arose, in part at least, from trying to conceive of logical notions and theorems in terms of universal algebraic concepts. Thus polyadic algebras (see Halmos [**Hal**]) and cylindric algebras (see Henkin, Monk, Tarski [**HMT**]) are algebraic versions of logic, and are algebraic structures (certain Boolean algebras with operators) which can be, and have been, studied in much the same way that one studies groups, rings, lattices, etc. At the same time that algebraic logic in this sense has been developing, category theory also developed, and in particular many facets of universal algebra were generalized (see Mac Lane [**Mc L**]). The interplay between category theory and logic may be considered to be another algebraization of logic. The present book gives one of the first systematic treatments of this kind of algebraization.