difficult." Indeed, we do. After studying this chapter, mathematicians will find these discussions less difficult.

Prerequisites for profitable reading of this book consist of a knowledge of basic graduate level differential geometry and a knowledge of Newtonian physics at least equivalent to a good freshman level course in the subject. The presence of many exercises make the book appropriate as a text, perhaps for example as the second semester of a first graduate course in differential geometry. The first half of the book should already sufficiently prepare the reader for entry into the physics literature with a minimum of trauma.

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Stochastic integration and generalized martingales, by A. U. Kussmaul, Pitman Publishing, London, San Francisco, Melbourne, 1977, 163 pp., $14.00.

Modern stochastic integration began in 1828 when the English botanist Robert Brown observed the motion of pollen grains in a glass of water. Bachelier (1900, [2]) and Einstein (1905, [7]) studied the mathematical modelling of the motion of the grains, now called *Brownian motion*. But it was Wiener (1923, [25]), making use of the ideas of Borel and Lebesgue, who created the modern rigorous mathematical model of Brownian motion, the *Wiener process*.

The Wiener process has three properties which make it of fundamental importance to the theory of stochastic processes: it is a Gaussian process, it is a strong Markov process, and it is a martingale. Let $W = (W(t, \omega))_{t \geqslant 0}$ denote a Wiener process, in which t is the time and each ω is a particle; then $W(t, \omega)$ represents the position of that particle at time t. One can show that except on a set of probability zero, every sample path (i.e., $W(t, \omega)$ as a function of t for fixed ω) is continuous but is of unbounded variation on every compact time set. Since the sample paths are nowhere differentiable, the Brownian particle cannot have an instantaneous velocity. This bizarre property may be viewed as a consequence of the Markov property; that is:

knowledge of both the present and past positions of the particle is no better at predicting its future location than knowledge of the present position alone. If a Brownian particle were to have instantaneous velocity then both the present position and the velocity would be needed to predict the future position; such a dependence on the past behavior of the particle would violate the Markov property.

Since the sample paths are of unbounded variation on every compact set, they cannot be differentials in a Stieltjes integral. Although Stieltjes integration with respect to the paths of the Wiener process is not possible, the differential dW does have an intuitive interpretation. Engineers think of dW as *white noise*, and by the use of generalized functions [1, pp. 51–53] one can define the quantity dW rigorously. Wiener (1933, [20]) gave meaning to dW in his definition of what is called the Wiener integral, but in such integrals the integrands are functions of time only (*certain* functions). It was K. Itô (1944, [9]) who first defined an integral for random integrands with respect to the Wiener process. Itô used his integral to represent a large class of diffusions as solutions of stochastic differential equations. This representation provides an intuitive and purely probabilistic construction of diffusions. The difference between ordinary calculus and the stochastic calculus developed by Itô is most strikingly illustrated by the famed Itô change of variable formula, valid for any f in C^2:

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds. \tag{1}$$

Of course, in ordinary Riemann-Stieltjes integration the third term on the right side of (1) does not appear. Itô's integral has the important feature that for appropriate integrands $H = (H(s, \omega))_{s \geqslant 0}$, the process $(\int_0^t H_s dW_s)_{t \geqslant 0}$ is a martingale, one of the fundamental properties of the Wiener process.

A *martingale* is a stochastic process $X = (X_t)_{t \geqslant 0}$ such that $X_t \in L^1$ for each t and such that the conditional expectation satisfies the relation

$$E(X_t|\mathfrak{F}_s) = X_s \qquad (s < t) \tag{2}$$

where $\mathfrak{F}_s$ is a σ-algebra representing all observable events before time s. Doob (1953, [6]) extended Itô's work on integration by using martingales instead of Wiener processes. The integral is so constructed that integration with respect to a martingale yields a martingale. If (2) is replaced by $E(X_t|\mathfrak{F}_s) \geqslant X_s$, the process is called a *submartingale*. In 1962 Meyer [16] found the right conditions under which a certain decomposition of submartingales is possible; Kunita and Watanabe (1967, [11]) then used this decomposition to reveal an elegant theory of stochastic integration with respect to martingales. Doléans-Dade and Meyer [5] removed an assumption on the underlying filtration of σ-algebras and proved a general change of variables formula, an extension of (1).

The Wiener process contributed to the development of stochastic integration in another, more circuitous fashion. Doob revealed a connection between the Wiener process and classical potential theory, and the fundamental papers of Hunt in 1956 helped to lay the groundwork of probabilistic potential theory. The Strasbourg school (Meyer, Dellacherie, and others) realized that many of the concepts developed in Markov process theory have significance when applied to stochastic processes that do not satisfy the Markov property. The basic underlying structure of a stochastic process $(X_t)_{t\geqslant 0}$ is a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where one assumes: (1) that $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leqslant t$; (2) $\mathcal{F}_0$ contains all P-null sets; and (3) $\bigcap_{t>s}\mathcal{F}_t = \mathcal{F}_s$. These three hypotheses are said to be the *usual hypotheses*. If X_t is $\mathcal{F}_t$-measurable for each $t \geqslant 0$, X is *adapted*. The process X customarily satisfies additional hypotheses; for example, it may be a Markov process, a point process, a martingale, or a Gaussian process. The *general theory of processes* is the study of stochastic processes which need satisfy only the usual hypotheses.

From the standpoint of stochastic integration the development of the general theory of processes has been crucial. One example is the use of the optional and predictable σ-fields. A stochastic process $(X_t)_{t\geqslant 0}$ is a function mapping $\Omega \times [0, \infty[$ into $\mathbf{R}$. There are several useful σ-algebras with which one can endow $\Omega \times [0, \infty[$. The most naive is $\mathcal{F} \otimes \mathcal{B}$, where $\mathcal{B}$ is the collection of Borel sets of $[0, \infty[$. For technical reasons σ-algebras that evolve with time are employed. The *optional σ-algebra* $\mathcal{O}$ is the smallest σ-algebra generated by adapted processes with *right* continuous paths. The *predictable σ-algebra* $\mathcal{P}$ is generated by adapted processes with *left* continuous paths. In general, $\mathcal{P} \subseteq \mathcal{O} \subseteq \mathcal{F} \otimes \mathcal{B}$. Among the deepest results of the general theory of processes are the *Section Theorems*. These theorems are used to prove the existence of a unique projection of a bounded $\mathcal{F} \otimes \mathcal{B}$-measurable process onto the optional and predictable processes.

In the case of the canonical Wiener process, the optional and predictable σ-algebras coincide. In addition, an adapted process which is $\mathcal{F} \otimes \mathcal{B}$-measurable comes very close to being optional, in a technical sense. Attempts were made to extend the Itô integral to a martingale integral which allowed $\mathcal{F} \otimes \mathcal{B}$-measurable, adapted processes as integrands. The general situation is complicated. If the integrands are restricted to *predictable* processes that satisfy appropriate finiteness conditions, then the resulting integral has the following desirable properties: (a) it is a martingale; (b) when the differential has paths of bounded variation, the integral agrees on those paths with a Lebesgue-Stieltjes integral; (c) letting $\Delta M_t = M_t - M_{t-}$ (the jump at t), the two processes $(\Delta\int_0^t H_s dM_s)_{t\geqslant 0}$ and $(H_t \Delta M_t)_{t\geqslant 0}$ are indistinguishable. Properties (b) and (c) fail to hold, in general, if predictable integrands are replaced by adapted and $\mathcal{F} \otimes \mathcal{B}$-measurable integrands, or even if optional integrands are used.

If all the paths of an adapted process are right continuous and of finite variation on compact time sets, we call the process a *VF process*. If V is a *VF* process and H is a bounded predictable process then, for each fixed ω, we denote by $\int_0^t H_s(\omega)\, dV_s(\omega)$ the Lebesgue-Stieltjes integral. There is no ambiguity in notation here because of property (b) of the martingale integral discussed previously. A stochastic process is a *local martingale* if certain integrability conditions in the definition of a martingale are relaxed. It turns out that the development of stochastic integrals can be extended to local martingales. A stochastic process X is a *semimartingale* if X can be written in the form

$$X = L + V \tag{3}$$

where L is a local martingale and V is a VF process. If H is a bounded, predictable process, one can then define $\int_0^t H_s\, dX_s$ by

$$\int_0^t H_s\, dX_s = \int_0^t H_s\, dL_s + \int_0^t H_s\, dV_s. \tag{4}$$

Although the decomposition (3) need not be unique, the integral $\int_0^t H_s\, dX_s$ in (4) is nonetheless well defined as a consequence of property (b) of the martingale integral. We will refer to the stochastic integral given by (4) as *the semimartingale integral.*

The semimartingale integral is perhaps the best known extension of the Itô integral, but it is not the only one. Fisk and Stratonovich [**24**] independently discovered a symmetrized integral for Wiener process differentials which obeys the rules of ordinary calculus. However one loses the property that the integral $(\int_0^t H_s\, dW_s)_{t \geqslant 0}$, when considered as a process, is a martingale. Meyer [**18**, p. 360] has proposed an extension of the Fisk-Stratonovich integral to semimartingale integrators. Unfortunately the class of integrands admissible for the Fisk-Stratonovich integral is much smaller than the class of admissible integrands for the semimartingale integral.

In his book on stochastic calculus McShane (1974, [**12**]; see also [**13**]) proposes a stochastic integral which also extends the Itô integral to more general differentials than the Wiener process. McShane's differentials in most cases must satisfy a technical requirement, which he calls the $K\Delta t$ conditions. It has been shown, however, that $K\Delta t$ processes are semimartingales, and the relationship between the McShane integral and the semimartingale integral has been determined (cf. Protter [**23**]).

The semimartingale integral as previously described is restricted to real-valued processes. Kunita [**10**], Metivier [**14**], [**15**], Pellaumail [**21**] and others have developed a theory of stochastic integrals which are Hilbert-space and Banach-space valued. This approach uses not only the general theory of processes, but also vector-valued measures. This development led to several nice martingale representation theorems in $\mathbf{R}^n$ (cf. Galčuk [**8**], Meyer [**19**]).

The book by Kussmaul reviewed here is the first introductory text on stochastic integration in English. Kussmaul takes the most general approach, that of vector-valued measures, although in the scalar case he shows in §11 the relationship of his development to the better known semimartingale integral. Kussmaul's presentation of the general theory of processes is streamlined. This may prove to be valuable to a newcomer to the subject, since Dellacherie's book [**3**] (in French), the standard reference for the general theory, contains the proofs of several difficult theorems which are not essential for an understanding of the basic principles of stochastic integration. It is regrettable, however, that Kussmaul has gone to the other extreme, and omitted the proofs of the section theorems, which are among the key results of the subject.

The stochastic integral is useful both as a theoretical and a practical tool. For example, the original proofs of the existence of a Lévy system for a strong Markov process used stochastic integrals (cf., e.g., Meyer [**17**]). When a real-valued Markov process Z is also a semimartingale, one can give rigorous meaning to the intuitive notion dZ by the use of stochastic integrals (cf. Protter [22]). On the applied side, stochastic integrals and stochastic differential equations have been used in stastistical communication theory. Kussmaul gives a taste of the usefulness of stochastic integration by proving a martingale representation theorem.

Extensions of the Itô integral to local martingales and to semimartingales have always been accompanied by proofs of the appropriate version of Itô's lemma (1). This lemma is not only essential for an understanding of the stochastic integral, but also for the calculation of closed-form expressions of many useful integrals. The proofs of a large number of results in the theory of stochastic integrals and stochastic differential equations require Itô's lemma (cf., e.g., Doléans-Dade [4]). Unfortunately, the book by Kussmaul, as he himself points out in the preface, fails to present a change of variables formula. Since Kussmaul does not give any reference to Itô's lemma, it is up to the reader to consult the literature. Good sources for this lemma are Metivier and Pistone [15], Meyer [18], or Pellaumail [21].

Although the book is introductory in nature, Kussmaul points out in the preface that the reader should have a knowledge of measure theory, some probability, and some Banach space theory. However, various parts of the book make other demands on the reader. Since Kussmaul concentrates on the vector-valued measure approach, which does not have nearly so well-developed a theory as does the scalar case, it is advisable that the reader have some familiarity with scalar stochastic integration. This is especially desirable because of the abstract nature of the approach. In fact, it is conceivable that an uninitiated reader of this book might not become aware of the differences between a martingale integral and a randomized Stieltjes integral.

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The theory of information and coding: A mathematical framework for communication, by Robert J. McEliece, Addison-Wesley, London, Amsterdam, Don Mills, Ontario, Sydney, Tokyo, 1977, xvi + 302 pp., $21.50.

In the beginning (30 years ago) were Shannon and Hamming, and they took two different approaches to the coding problem. Shannon showed that the presence of random noise on a communications channel did not, by itself, impose any nonzero bound on the reliability with which communications could be transmitted over the channel. Given virtually any statistical description of the channel noise, one could compute a number C, called the channel capacity, which is a limit on the rate at which information can be transmitted across the channel. For any rate $R < C$, and any $\varepsilon > 0$, one could concoct codes of rate R which would allow arbitrarily long blocks of information to be transmitted across the noisy channel in such a way that the entire block could be correctly received with probability greater than $1 - \varepsilon$. Shannon's results were astounding and, at first, counterintuitive. However, they opened an area of study which has continued until this day. Modern practitioners of the "Shannon theory" continue to study questions of what performance is theoretically possible and what is not when one is free to use asymptotically long codes. The major activity in this area in the last few years has been related to questions about networks of channels, and broadcast channels, in which the same transmitted information is corrupted by different types of noise before being received by many different receivers. The main