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BOOK REVIEWS

The new elements of mathematics, by Charles S. Peirce, Carolyn Eisele (editor), Humanities Press, Atlantic Highlands, N.J., 1976, cxxxviii + 2478 pp., $283.00. Four volumes in five books:
Vol. I, *Arithmetic*, xl, 260 pp., $31.00.
Vol. II, *Algebra and Geometry*, xxxi, 672 pp., $76.00.
Vols. III/1, III/2, *Mathematical Miscellanea*, xxix, 1153 pp., $128.00.
Vol. IV, *Mathematical Philosophy*, xxviii, 393 pp., $48.00.

Charles S. Peirce, 1839 to 1914, was one of America's most outstanding intellects. Philosopher, mathematician, and scientist, he wrote profusely, the equivalent of almost 100,000 printed pages in all. He was able to publish only about one-seventh of this, and most of his writings in mathematics and logic were never published during his lifetime.

The *Collected papers of Charles Sanders Peirce* appeared from 1931 to 1958; volumes 1 to 6 were edited by Charles Hartshorne and Paul Weiss (1931–1935), and volumes 7 and 8 by the present writer (1958). While these volumes included some previously unpublished papers in mathematical logic, by design they included almost none of Peirce's other papers in mathematics, nor his drafts of textbooks.

Carolyn Eisele, Professor Emeritus of Mathematics at Hunter College, has now filled this gap. She has edited about 2500 pages of the unpublished manuscripts, encompassing pure mathematics, numerous applications, and some rather ingenious textbook materials. *The new elements of mathematics* includes Peirce's papers on linear algebra and matrices, Euclidean and non-Euclidean geometry, topology and Listing numbers, graphs, and the four-color problem; also, his mathematical applications to economics, map projections, engineering, and the theory of errors. In addition, there are writings on the logic of relatives, Boolean algebra, and the nature of continuity; on probability, inductive logic, and applications of induction to historical inquiry. Finally, Professor Eisele provides most of Peirce's drafts of textbooks on arithmetic, geometry, and trigonometry.

Charles Peirce was the son of Benjamin Peirce (1809–1880), America's first original mathematician, whose *Linear associative algebra* appeared in 1870. Charles derived many results from his father's algebras, and he demonstrated their connection to relations (matrices).

Charles Peirce also proved a theorem about the rotation of bodies in four-dimensional space. But his most important mathematical results were in symbolic logic, a subject not generally accepted by mathematicians in Peirce's time. He developed the formalism of the propositional calculus and the general logic of quantifiers, independently of, though a little later than, Gottlob Frege. Independently of Dedekind, Peirce defined a finite set as one that cannot be put in one-one correspondence with a proper subset of itself.

He was the first to define all Boolean functions in terms of the single primitives "not-and" and "not-or" and to conceive of the truth-table method of evaluating truth-functions. He also was the first to develop multi-valued logics.

Peirce's only regular occupation was as a physicist with the United States Coast and Geodetic Survey. He achieved great distinction in this position, and, appropriately, the National Oceanic and Atmospheric Administration now has an instrumented ship named after him. Peirce's job was to measure and compare the force of gravity at various places in the United States and Europe by swinging pendula, to help determine the ellipticity of the earth. He became fascinated by the problem of experimental error and its relation to the classical issue of determinism versus indeterminism. Using both experiment and statistical theory, he analyzed the errors in measuring the force of gravity with the current pendula and techniques, and proved that the errors were 100 times as great as the experts had believed!

His interest in the accuracy of measurement led Peirce to suggest that the best standard of length would be a wavelength of light produced by a specific element. Such a standard is now used, the meter being defined as so many wavelengths in vacuum of the orange-red line of the spectrum of a particular isotope of krypton.

Charles Peirce's most original work was in general logic, the methodology of inquiry, and the philosophy of science. He was interested in applying mathematics to inquiry in many forms and in many areas. He was the first to develop inductive logic in terms of statistics and probability, and the first to formalize modal logic. He sought a quantitative measure of information. He connected statistics to Darwin's theory of the evolution of natural species and was the first to see that evolution could be given a mathematical formulation. He used calculus to analyze the market, and made a mathematical analysis of the most economic way to do research. He wrote a long paper on the inductive logic of history.

When Peirce worked in science and mathematics he was constantly thinking of their methods, their foundational principles, and their philosophical implications. He founded pragmatism, developing it out of his probabilistic analysis of the evolution of belief. He created an evolutionary metaphysics embracing cosmic evolution, biological evolution, and social evolution. According to Peirce, the basic laws are probabilistic and not reducible to universal causal laws. These laws produce change gradually, so that mathematically continuous functions are always applicable.

The role of these two factors of chance and continuity in Darwinian evolution is clear enough. Chance produces the variation to be tested by the environment, and the evolution of species is continuous, not discrete as in creationism. It is interesting to see how Peirce's two principles have fared in quantum mechanics, a subject that appeared after his death. Let us assume that John von Neumann's foundational analysis holds for the ultimate form of quantum mechanics. Then, by his theorem that there are no hidden parameters, the probabilistic laws of quantum mechanics are basic and are not reducible to deterministic laws. So Peirce is correct on the basic role of chance. But since matter as depicted by quantum mechanics is discrete, Peirce's continuity principle fails.

Although Peirce never held a regular university position, he did teach for a few years at Johns Hopkins in Baltimore, while still working for the Coast Survey in Washington. He had some excellent students, whom he taught methods of research: Joseph Jastrow, who did experimental work with Peirce on the probabilistic character of the threshold of sensation; Christine Ladd-Franklin, who invented a new way of testing syllogisms and later a theory of color vision; and Alan Marquand, who designed logic machines and later became an outstanding professor of classical archeology at Princeton University. John Dewey was also a student of Peirce's at Johns Hopkins, but was not influenced by him until later.

In his opening lecture to his logic course in the fall of 1882, Peirce said:

> This is the age of methods; and the university which is to be the exponent of the living condition of the human mind, must be the university of methods . . .
>
> . . . when new paths have to be struck out, a spinal cord is not enough; a brain is needed, and that brain an organ of mind, and that mind perfected by a liberal education. And a liberal education–so far as its relation to the understanding goes–means *logic*. That is indispensable to it, and no other one thing is.

Peirce was advocating two interrelated policies here. First, that the university should teach its students how to solve problems and obtain new results. The student's acquisition of existing knowledge was to be directed toward this end. Peirce himself successfully employed this policy with Jastrow, Ladd-Franklin, and Marquand.

Peirce's second theme was that science was becoming aware of its methodology and that new methods of research were being developed, such as experimental method in psychology. Further, Peirce thought the time was ripe for the general theory of methods, the core of which was logic in a sense both broad and deep. He regarded mathematical logic as the foundation of all reasoning. Inductive logic, including the applications of probability and statistics, was a second important branch of logic. The third branch was a logic of discovery that would contain rules and procedures helpful in solving problems and discovering new results. Peirce believed that his general theory of methods would unify the methods employed in inquiry and lead to their improvement.

About twenty years later Peirce sought to apply his conception of methodology and its role in education to the teaching of mathematics. He wrote textbook manuscripts in arithmetic, algebra, and geometry. These drafts are among the most interesting papers in *The new elements of mathematics*; they are reproduced in volumes I and II.

To place Peirce's textbooks in proper historical perspective, we should bear in mind that the teaching of mathematics in his time relied heavily on rote learning and extended practice, with little emphasis on intuitive understanding or the interrelations of the concepts and rules or the relation of mathematical entities to their instances and applications. This general situation continued, with gradual improvement, until about twenty years ago, when the Russians, with their Sputnik, beat the United States in the race to put a satellite in space. That event led to a revolution in the teaching of elementary and intermediate mathematics. Some of the methods developed were strikingly similar to those Peirce had advocated, unsuccessfully, nearly sixty years earlier.

I recall, as a mathematics student in the thirties, that set theory and logic did not appear in the curriculum until graduate school. After Sputnik, the stress shifted from rote computational skills to teaching concepts. Some set theory was introduced in elementary school and some logic in high school.

The pedagogy of mathematics was improved considerably. But there were defects. Sometimes too much conceptual apparatus was introduced in the lower grades, with too little attention to computation and applications; bright high school students were sometimes pushed beyond their maturational capacities. And subject-matter change, however desirable, could not remedy the scarcity of able and inspiring teachers, more important in mathematics, perhaps, than in any other discipline.

Let me illustrate the problem at the elementary level with an extreme example that came to my attention. The teacher asked the pupils to "get out their sets," at which they removed strings and counters from their desks. She next asked them to form "the null set," whereupon they arranged their strings in empty circles! She then asked them to form the set for one, whereupon they put a counter inside the circle; and similarly for two, three, etc.

Peirce had a better way, in my opinion, of teaching counting and the basis of set theory to young children. He considered ordinals psychologically more basic than cardinals. To combine concepts and applications, he used decks of sequentially-numbered cards. The teacher placed some of these cards face down. The question, "How many of the cards are turned face up?" brought the answer, "None." Then the cards were turned up one by one and the question repeated, with the children counting aloud. The cards were also arranged in various patterns, with questions asked to stimulate the manipulation of numbers and familiarity with their systematic character. That the children were working with ordinal sets was a mathematical concept to be introduced later, when the students were more mature.

For more advanced students Peirce explained arithmetic operations in terms of counting the elements of sets in the following way:

> *Addition* is the operation of finding how many in all there are in two or more mutually exclusive collections.
>
> *Multiplication* is the operation of finding how many *pairs* there are of which one member comes from one and the other from another collection. Continued multiplication finds how many *sets* there are of which one member is drawn from a collection of given *quotal* number. Thus, twice three is

Figure 1 (6 ways)

> *Involution* is the operation of finding in how many ways every member of one collection can be paired with a member of another collection. Thus 2^3 is as follows

Figure 2 (8 ways)

while 3^2 is as follows

Figure 3 (9 ways)

(Vol. I, *Arithmetic*, pp. xxxiv–xxxv.) He treated other bases than ten, and he had an interesting algorithm for quickly adding lists of binary numbers. He introduced binary numbers with the game of twenty questions.

Peirce tried hard to get his textbooks published, and he even had the assistance of his brother, James Mills Peirce, a Harvard professor of mathematics. He failed, partially because he, like many original minds, could teach and inspire the ablest, but could not reach average students and average teachers. But a more basic reason for Peirce's failure was that he was too far ahead of his time.

Of course, it is a manifestation of genius to have an idea long before it is understood and appreciated. Let me close by outlining the background for another of Peirce's logical ideas of great originality, the idea for a general-purpose relay computer, which was fifty years ahead of its time. The sequence of events is as follows:

1. Peirce stimulated Alan Marquand to invent and build a mechanical logic machine superior to that of William Stanley Jevons. This machine is described in Peirce's *Logical machines*, vol. III, pt. 1, pp. 625–632.

2. This machine was built in the early 1880s. At about the same time, Peirce conceived the sufficiency of "not-and" and "not-or," together with the use of a truth-table as a decision procedure for tautologyhood.

3. In a letter to Marquand dated 1886 Peirce suggested the use of relays for Marquand's machine and showed how to achieve "and" and "or" with relays. "... it is by no means hopeless ... to make a machine for really very difficult mathematical problems (*ibid.*, p. 632).

4. Marquand then prepared a wiring diagram for a relay version of his mechanical logic machine.

5. In 1900, Peirce stated that a computer could enumerate all the theorems of axiomatic arithmetic, thus anticipating the 20th century identification of logic with computers. See *Our senses as reasoning machines*, vol. III, pt. 2, pp. 1114–1115.

Peirce knew of Charles Babbage's attempt to build an "analytical engine." This was to be a general-purpose mechanical computer for calculating functions and making tables. Babbage worked on his machine a long time, but never completed it, partly because of the inadequacy of mechanical technology for that purpose. I think that when Peirce wrote his 1886 letter he saw that a relay version of Babbage's machine could be built and that it would work. The first general-purpose relay computer was completed after World War II, at about the same time that the ENIAC, the first general-purpose electronic computer, was completed!

We are all greatly indebted to Professor Carolyn Eisele for bringing together the most important of Peirce's unpublished manuscripts on mathematics; for her discriminating selection from a vast amount of material; and for her extensive historical researches, the results of which she has presented in the introduction.

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Supercompactness and Wallman spaces, by J. Van Mill, Mathematical Centre Tracts 85, Mathematisch Centrum, Amsterdam, 1977, iv + 238 pp.

Supercompactness spaces are compact spaces characterized by having a binary subbase. That is, there is a subbase $\mathcal{S}$ for the closed sets with the property that if $\mathcal{L} \subset \mathcal{S}$ and $\bigcap \mathcal{S} = \varnothing$ then there are two sets $L_0, L_1 \in \mathcal{L}$ such that $L_0 \cap L_1 = \varnothing$. The main goal of this monograph is to study supercompact spaces and supercompact extensions of arbitrary topological spaces.

Supercompact spaces were first defined by J. DeGroot in 1967 and arose from investigations on complete regularity and compactification theory. A number of mathematicians became interested in DeGroot's work and many of his conjectures have now been proved, new techniques have been developed, and new questions have arisen. This book presents known and new results in a structured form with sufficient auxiliary and background material to make the subject accessible to a reader with a solid course in graduate level general topology. The main appeal of the book however will be for those who have an interest in pursuing research in this area.

In the first chapter, supercompact spaces are studied in general. Topics included are: Hausdorff continuous images of supercompact Hausdorff spaces, the notion of an interval structure and its use in characterizing supercompactness, the relation between graphs and supercompact spaces, regular supercompact spaces (those possessing a binary base which generates a ring of regular closed sets), and partial orderings on supercompact spaces. DeGroot conjectured that every compact metric space is supercompact and that not every compact Hausdorff space is supercompact. Although these conjectures have been proven true, there are still many open questions and several are explicitly mentioned.

Using the notion of maximal linked systems, supercompact extensions of topological spaces are obtained in a manner analogous to the construction of Wallman-type compactifications. In Chapter II, properties of superextensions (and their subspaces) are studied: those they inherit from the underlying space, and those which are new and unexpected. The contractibility of superextensions is also investigated.

Metrizable superextensions are studied in Chapter III, with particular emphasis on infinite dimensional problems, such as: is the superextension of the closed unit interval homeomorphic to the Hilbert cube? (An affirmative answer is given.)

The subject of Chapter IV reflects the second part of the book's title and relates only incidentally to supercompact spaces. Two questions are