

extended to the case where there is learning. It is important to consider the fuzzy versions of these problems. The word "fuzzy" is used in the sense of Lotfi Zadeh.

Finally, a historical note. Problems of this type were considered by many mathematicians: Euler, Hamilton and Steiner, to name a few. But the first systematic study of these problems was carried out at the RAND Corporation during the years after 1948 under the inspiration, and often participation, of von Neumann. Major names were: George Dantzig, Stuart Dreyfus, Lester Ford and Ray Fulkerson. Many other mathematicians worked on these problems. They are closely connected with the theory of games, linear and nonlinear programming, as well as integer programming.

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Doubly stochastic Poisson processes, by Jan Grandell, Lecture Notes in Mathematics, vol. 529, Springer-Verlag, Berlin, Heidelberg, New York, 1976, x + 234 pp., \$10.30.

A point process on a space T is a random distribution of points throughout T . Its values are atomic measures on the space with the atoms having weights 1, 2, 3, . . . (corresponding to 1, 2, 3, . . . points at the atoms). An ordinary stochastic process refers to a random entity whose possible values are functions. In contrast, a point process refers to a random entity whose possible values are counting measures. Inherent and central to the notion of such a process is the idea of whether the points tend to be abundant and closely packed or sparse and widely separated, i.e. their intensity. To formalize this idea, suppose I is a measurable subset of the space. Suppose $N(I)$ denotes the number of points that are in I for a realization of the process. Then the expected or average value of $N(I)$, $E\{N(I)\} = \mu(I)$, is called the intensity measure of the process. In the case that the space is the real line, and the measure μ is absolutely continuous, its derivative is called the intensity function of the process. A point process is called Poisson with intensity measure μ if (i) for measurable I ,

$$(1) \quad \text{Prob}\{N(I) = n\} = \mu(I)^n \exp\{-\mu(I)\} / n!,$$

$n = 0, 1, 2, \dots$, and (ii) for measurable and disjoint I_1, \dots, I_K , the variates $N(I_1), \dots, N(I_K)$ are statistically independent, $K = 2, 3, \dots$. Such a process always exists when the space is locally compact with countable basis. A surprising result of Renyi [7] indicates that a point process with isolated points is necessarily Poisson with intensity measure μ if (1) holds for $n = 0$ and all bounded I in an algebra containing a basis for the topology of the space. Suppose further that the measure μ is itself random, then N is called the doubly stochastic process corresponding to μ . One has, for example,

$$(2) \quad \begin{aligned} & \text{Prob}\{N(I_1) = n_1, \dots, N(I_K) = n_K\} \\ &= E_\mu \left[\prod_{k=1}^K \mu(I_k)^{n_k} \exp\{-\mu(I_k)\} / n_k! \right]. \end{aligned}$$

As the definition suggests, the theory of these processes is close to that of stochastic measures. Doubly stochastic Poisson processes were apparently first introduced in Quenouille [6] and Cox [3].

There are a number of physical phenomena for which the doubly stochastic Poisson appears to provide an especially appropriate model. To mention one: light incident upon a sensitive photodetector results in an output signal made up of near identical pulses of brief duration relative to the gaps between. These pulses correspond to the emissions of individual photoelectrons by the detector. Point process data results from ascribing to the output only the "times" of the pulses. The existing theory of photodetectors, see for example Rousseau [8], strongly suggests that for a given input signal the output process will be Poisson with intensity function proportional to the instantaneous incident light intensity. Now, in many circumstances there are good reasons for viewing the light intensity as random and, hence, one ends up dealing with a doubly stochastic Poisson process here. (It is worth mentioning though that although one continually sees stated the "fact" that radioactive emissions provide a good example of Poisson process data, Berkson [1] analyses a lengthy stretch of data that does not appear to bear this out.)

Previous to now books have appeared with chapters devoted to the doubly stochastic Poisson (for example Cox and Lewis [4], Srinivasan [10], Snyder [9]). With the appearance of this work of Grandell, there is a monograph on the topic. This literature no doubt reflects the predominant position the process has amongst point processes when it comes to a need for specific results, in the same way that the Gaussian process is dominant amongst ordinary stochastic processes.

Readers are assumed to have a basic knowledge of the theory of stochastic processes, but the author emphasizes neither theory nor practice. The monograph has seven sections (really chapters) and an appendix containing certain required technical results. The monograph concentrates on the case where the basic space $T = (-\infty, \infty)$. Among other things: §1 presents

several alternate definitions of a doubly stochastic Poisson process, §2 considers special models and results conditional on an event having occurred at a specific time, §3 considers random measures, §4 presents limit theorems, §5 is concerned with linear and nonlinear estimation of related random variables, §§6, 7 consider the case of $T = \{0, \pm 1, \pm 2, \dots\}$ and, in particular, the estimation of the covariance function of a stationary random intensity. The results of a number of simulations are presented. None of the topics is pushed overly far. There is some discussion of the conditional intensity function (cf. Brémaud [2]) arousing such great interest these days, but not a lot. An index of notation would have helped this reader considerably.

One warning must be made relating to the use in practice of the doubly stochastic Poisson. Because it is *doubly* stochastic, its inherent randomness is greater than that of a Poisson process. There exist, however, numerous situations where this overdispersion does not occur, for example, situations wherein the occurrence of a point is tending to inhibit the occurrence of nearby points. An interesting problem is to produce general models that can handle such situations. A further interesting question involves describing the random *signed* measures μ that lead to doubly stochastic Poissons in a formal manner, i.e. satisfying (2). Milne and Westcott [5] consider the case of Gaussian μ .

The monograph is a definite contribution to the literature of point processes. We may expect to see many more on the topic.

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