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Estimates for the $\bar{\partial}$*-Neumann problem*, by P. C. Greiner and E. M. Stein, Princeton Univ. Press, Princeton, N. J., 1977, 194 pp., $6.00.

Several complex variables has enjoyed a renaissance in the past twenty-five years, reaching deeply into modern algebra, topology, and analysis for techniques to attack long standing problems. An important example is the question of identifying domains of holomorphy, i.e. those open sets in $\mathbf{C}^{n+1}$ (or, more generally, in complex manifolds) for which at least one holomorphic function has no extension outside the set. Early in this century E. E. Levi defined a condition, now called *pseudoconvexity*, which he proved was necessary, and conjectured was sufficient, to characterise domains of holomorphy. More precisely, for domains with smooth boundary one can define a Hermitian form, now called the *Levi form*, on the space of holomorphic vectors tangent to the boundary. The domain is then called *pseudoconvex* (resp. *strictly pseudoconvex*) if the Levi form is positive semidefinite (resp. definite).

Levi's conjecture for $\mathbf{C}^{n+1}$ was finally proved nearly fifty years later by Oka [16] (and simultaneously by Bremermann, [1] and Norguet [15]) after a long series of related papers. Efforts to extend the results to complex manifolds led Grauert [5] to discover a new, more general proof making extensive use of sheaf theory. A totally different proof was later obtained by Kohn [11] (using a crucial estimate of Morrey [13]) as a consequence of his solution of the "$\bar{\partial}$-Neumann" boundary value problem in partial differential equations.

Since Kohn's breakthrough on the problem there has been considerable interest in constructing solutions for the inhomogeneous Cauchy-Riemann (C-R) equations in a bounded complex domain and studying their boundary behavior. Kohn's methods, based on a priori L^2 estimates, give only L^2 existence proofs for solutions of the C-R equations. (After Kohn's work appeared Hörmander [9] gave a simpler existence proof, using weighted L^2 estimates, in which boundary problems are completely circumvented!) Several explicit solutions have been constructed by the use of integral formulas, in particular, those of Henkin [8] and Ramirez [18]. Kerzman [10], Grauert and Lieb [6], Overlid [17], and others have obtained estimates for these solutions in terms of L^p and Lipschitz norms.

Recently Greiner and Stein were able to give an explicit construction of Kohn's solution and to obtain from this construction optimal estimates in L^p and other norms. The book under review is an exposition of this work, announced in [7]. The techniques involved are taken from the theory of singular integrals, including operators on nilpotent Lie groups, as well as from the theory of "classical" psuedodifferential operators. Although the proofs may appear to be highly computational, the main ideas behind them can be stated in terms of elegant principles. The reader who is familiar with the subject should read the excellent but concise introduction where these principles are enunciated. For other readers this review is intended as an introduction to that introduction.

The $\bar{\partial}$-Neumann problem was proposed by D. C. Spencer in the early '50s as an approach to studying the inhomogeneous Cauchy-Riemann equations

$$\partial U/\partial \bar{z}_j = f_j, \qquad j = 1, 2, \ldots, n+1,$$

for $\{f_j\}$ satisfying the compatibility conditions

$$\partial f_j/\partial \bar{z}_k = \partial f_k/\partial \bar{z}_j, \qquad 1 \leqslant j, k \leqslant n+1,$$

in a strictly pseudoconvex bounded domain $M = \{z \in \mathbf{C}^{n+1}: \rho(z) < 0\}$, ρ a smooth function.

In particular, suppose the f_j extend to smooth functions in a neighborhood of the closure $\bar{M}$ of M. (This extension property is written $f_j \in C^\infty(\bar{M})$.) Then there should be a solution $U \in C^\infty(\bar{M})$. In Spencer's approach, one rewrites the above system as a single equation on forms: Let $C^\infty_{0,q}$ be the space of smooth $(0, q)$ forms, i.e.

$$C^\infty_{0,q} = \left\{ \sum_Q f_Q \, d\bar{z}^Q, d\bar{z}^Q = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}, |Q| = q \right\}.$$

If $\bar{\partial}_0: C^\infty_{0,0} \to C^\infty_{0,1}$ is defined by

$$\bar{\partial}_0 g = \sum_{j=1}^{n+1} \frac{\partial g}{\partial \bar{z}_j} \, d\bar{z}_j;$$

and the definition extended to $\bar{\partial}_q: C^\infty_{0,q} \to C^\infty_{0,q+1}$, then the above system is equivalent to $\bar{\partial}_0 U = f$, with $f = \sum f_j \, d\bar{z}_j$ satisfying $\bar{\partial}_1 f = 0$. By choosing a metric on $\bar{M}$, one can define adjoints $\bar{\partial}_q^*$ and a Laplacian $\Box$ on $C^\infty_{0,1}$ by $\Box = \bar{\partial}_1^* \bar{\partial}_1 + \bar{\partial}_0 \bar{\partial}_0^*$. The equation $\Box u = f$ leads to the following boundary conditions on u: u is in the domain of $\bar{\partial}_0^*$ and $\bar{\partial}_1 u$ is in the domain of $\bar{\partial}_1^*$. The $\bar{\partial}$-Neumann problem is to prove existence and regularity for this boundary value problem. If $\bar{\partial}_1 f = 0$, then $U = \bar{\partial}_0^* u$ is a solution of $\bar{\partial} U = f$ which is orthogonal to the space of homomorphic functions in M.

Kohn solved the problem by proving the existence of an operator N, called the Neumann operator, which inverts $\Box$; i.e., $N\Box = \Box N = I$, where I is the identity operator. Furthermore, N maps compactly supported forms in $C^\infty_{0,1}(\bar{M})$ to $C^\infty_{0,1}(\bar{M})$, and the following estimate holds on Sobolev spaces:

$$\|Nf\|_{L^2_{k+1}} \leqslant C\|f\|_{L^2_k}, \qquad C > 0.$$

Here

$$L^p_k = \{f \in L^p: \partial^\alpha/\partial x^\alpha f \in L^p, \text{all } |\alpha| \leqslant k\}$$

with norm

$$\|f\|_{L_k^p} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f \right\|_{L^p}.$$

In the present work the authors explicitly construct the Neumann operator and obtain sharp estimates for solutions of $\Box u = f$ (and hence $\bar{\partial} U = f$) in terms of various function spaces. In particular, they prove the estimate $\|Nf\|_{L_{k+1}^p} \leq C\|f\|_{L_k^p}$ for $1 < p < \infty$, all $k \geq 0$. An L^p estimate for $U = \bar{\partial}_0^* u$ may then be obtained from this. In terms of Lipschitz space Λ_α, U is shown to satisfy $\|U\|_{\Lambda_{1/2}} \leq C\|f\|_{L^\infty}$. Examples show these estimates to be sharp. Further results show that derivatives of u and U in certain "allowable" directions enjoy greater smoothness than arbitrary derivatives.

As the reader proceeds to the construction of N, he will soon notice that all operators involved are dealt with in terms of their symbols. Recall that the symbol of an operator A: $C_0^\infty(\mathbf{R}^m) \to C^\infty(\mathbf{R}^m)$ is a smooth function $a(x, \xi)$ such that

$$Af(x) = \frac{1}{(2\pi)^m} \int e^{i\langle x, \xi\rangle} a(x, \xi) \hat{f}(\xi)\, d\xi.$$

A is a (classical) *pseudodifferential operator* of order k if the estimate

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} a(x, \xi) \right| \leq C_{\alpha,\beta} (1 + |\xi|)^{k - |\beta|}$$

holds for all multi-indices α, β, with x restricted to a compact set. (See e.g. [**14**].) The reader should be warned that not every operator whose symbol is calculated is actually a pseudodifferential operator. (G_α of Chapter 1 is not, for example.) Nevertheless, the theory of pseudodifferential operators enters in a crucial way. First, any elliptic pseudodifferential operator (i.e. one whose symbol is invertible, in a suitable sense, for $\xi \neq 0$) has an "inverse". Since $\Box$ is an elliptic operator in the interior of M an inverse can be found in the algebra of pseudodifferential operators. Next, by a method going back to Calderón [**2**], the $\bar{\partial}$-Neumann boundary conditions on U may be expressed as a pseudodifferential on the boundary ∂M of M.

To understand this reduction to the boundary, it may help to consider a much simpler example where calculations can be easily carried out. Let $\mathbf{R}_+^3 = \{(x, y, t): t > 0\}$ and let $g(x, y)$ be a smooth function with compact support defined on $\{t = 0\}$, the boundary of $\mathbf{R}_+^3$. Suppose now that $\eta = a\partial/\partial x + b\partial/\partial y + c\partial/\partial t$ is a given real vector field. The problem then is to find a function $v(x, y, t)$ harmonic in $\mathbf{R}_+^3$ (i.e. $\Delta v = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial t^2)v = 0$ in $\mathbf{R}_+^3$) such that $v \in C^\infty(\bar{\mathbf{R}}_+^3)$ and the restriction of ηv to $\{t = 0\}$ is g. If PI is the Poisson integral for $\mathbf{R}_+^3$, then $v = PI(v_b)$ where $v_b(x, y) = v(x, y, 0)$. Then

$$g(x, y) = \frac{1}{(2\pi)^2} \left[\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t} \right) \int_{\mathbf{R}^2} e^{i(x\xi_1 + y\xi_2) - t|\xi|} f(\xi_1, \xi_2)\, d\xi \right]_{t=0}$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} e^{i(x\xi_1 + y\xi_2)} (ia\xi_1 + ib\xi_2 - c|\xi|) \hat{f}(\xi)\, d\xi.$$

Thus $g = Av_b$, where A is a pseudodifferential operator on $\mathbf{R}^2$ with symbol

$ia\xi_1 + ib\xi_2 - c|\xi|$. To determine v_b, and hence v, it suffices then to invert A. If $c \neq 0$, i.e. if η is not tangential at any point, then A is elliptic and hence invertible.

In the case of the $\bar{\partial}$-Neumann problem the computation of the symbol of the boundary operator requires two chapters (7 and 8) of clever calculations. At the end of this arduous climb the reader may feel ill-rewarded to discover that the pseudodifferential operator obtained, denoted $\square^+$, is not elliptic. However, this is to be expected since the $\bar{\partial}$-Neumann problem is a nonelliptic boundary value problem. Were it elliptic it would have been solved years ago and this book would not have been written now! Since $\square^+$ does not have an inverse in the algebra of classical pseudodifferential operators one must look elsewhere. First the authors construct an operator $\square^-$ such that $\square^+\square^- = \square^-\square^+ = -\square_b$, where $\square_b$ is the "boundary Laplacian" of $\bar{\partial}$. (The above equalities hold modulo certain "acceptable" error operators. Also involved is an army of cut-off functions, which we suppress here.) Now if $-K$ is an operator which inverts $\square_b$, then $K\square^-$ is an inverse for $\square^+$. Hence it would be sufficient to invert $\square_b$.

The operator $\square_b$, which acts on (0, 1) forms on the boundary, has been carefully studied. For $n > 1$, Kohn [11] proved that $\square_b$ is hypoelliptic, i.e. if $\square_b w = g$ with g smooth in an open set, then w is also smooth in that set. Folland and Stein [4] (see also [19]) constructed an "inverse" for $\square_b$ when $n > 1$, and this is where singular integral operators on nilpotent groups enter the picture. The Heisenberg group with appropriate dilative automorphisms is a local model for the boundary. An inverse for $\square_b$ on the Heisenberg group is given by group convolution with a homogeneous function. A calculation of this function and of the symbol defining the inverse operator is given in Chapter 1. Going from the model case to the general one requires, among other things, a system of local group coordinates at each point in the boundary and a careful accounting of all errors resulting from this approximation by the Heisenberg group.

To study estimates of the solution u, after the Neumann operator has finally been constructed, the whole process must be re-examined for a closer look at the operators used in the construction. The authors provide a brief summary in Chapter 4 of the four types of operators involved. The action of each of these operators on various function spaces is then studied and estimates for the Neumann operator in these spaces are obtained. From these, the estimates for $\bar{\partial}_0$ are easily derived. Finally, the estimates for Lipschitz spaces are shown to be optimal by counterexamples.

This last part of the book contains a considerable amount of useful background material on known results. It includes, in particular, material on Lipschitz and Besov spaces and the boundedness properties of pseudodifferential operators on these spaces. Since these results are not easily available elsewhere, readers who are interested in these topics might do well to turn first to Part III.

It should be obvious that this book is not intended as an easily digested but superficial survey of recent results in several complex variables. However, for the serious reader the prerequisites are not as foreboding as they may seem. Although one should probably look at [4], where operators on the Heisenberg group are discussed, no other knowledge of nilpotent Lie groups is needed. Furthermore, most of the book uses only the very basic elements of the theory of pseudodifferential operators. With this background, and some determination, the reader can get through this carefully written exposition of important new results.

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