"little more than a plaything". The issue was forced by Bertrand's paradox, which arose from the fact that the measure is not unique for geometric events depending on continuous parameters. Poincaré answered this question neatly by requiring the measures to be invariant under a group of transformations. In almost all practical cases this defines the measure up to a constant factor. Poincaré also realized the importance of the kinematic measure, which nowadays is better understood as the Haar measure of a unimodular Lie group, and exploited it.

The basic reason for integral geometry is the presence of a "duality" in most spaces. Examples are points and lines in the plane, points and geodesics in a Riemannian manifold, points and lattices in R^n, points and horospheres in a symmetric space, etc. The two dual geometric elements are related by a notion of incidence. Given a set in space, the measure of the set of dual elements incident to it gives an important invariant of the set. The classical and simplest example is Crofton's theorem: The measure of the set of lines in the plane meeting an arc β, counted with multiplicities, is equal to twice the length of β. When the same idea is applied to lattices, it gives Siegel's proof of Hlawka's solution of a problem of Minkowski on convex bodies. Radon treated the problem of determining a function on the noneuclidean plane from the integrals of the function over all geodesics. This Radon transform was generalized by Fritz John, and later by Helgason, Gelfand, Graev, Vilenkin, and most recently by Guillemin. It plays an important role in partial differential equations and more general integral operators.

A natural application is to stereology, which deals with a body of methods for the exploration of three-dimensional space when only two-dimensional sections through solid bodies or their projections are available. Clearly stereology is useful in biology, mineralogy, and metallurgy. Recently the ideas and tools of stochastic processes are introduced, bringing the subject back to probability theory.

The strides made in the last four decades are enormous. Integral geometry is no longer a mathematical discipline to be ignored. But the subject has still the happy character that it is not so well known, thus allowing a steady and gentle progress.

As the first volume of an ambitious encyclopedia, the book sets a style. It is a combination of a lucid exposition of the introductory aspects and a complete survey of the area. No topic seems to have been left uncovered. There is also a very complete, but selective, bibliography. The author handled his material with great dexterity and ease. The book should serve as an excellent text for a graduate course on integral geometry. The encyclopedia and the author are to be congratulated for their success.

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Methods of accelerated convergence in nonlinear mechanics, by N. N. Bogoljubov, Ju. A. Mitropoliskii and A. M. Samoilenko, Hindustan Publishing Corporation, Delhi, India, viii + 291 pp., $27.90.

(I) The averaging method–or what is so called today–has its origin in celestial mechanics. It was already used by C. F. Gauss when he approximated the motion of the planets by rigid elliptical rings with an appropriate mass distribution and averaged the forces between these planets over these rings. The resulting equations describe in first approximation the changes of the ellipses under the influence of the perturbation. The later monumental work of Poincaré in celestial mechanics contains series expansions which refine the averaging method, in the frame work of Hamiltonian mechanics. Only considerably later, in 1934, did Krylov and Bogoljubov turn to applying these techniques to arbitrary systems which need not be Hamiltonian, and thus made the averaging method accessible to the down-to-earth mechanics in which some friction is always present, making the equation nonconservative. For example, oscillations of nonlinear electric circuits could be successfully approximated by the averaging method.

This method has primarily been useful in the description of periodic motions with one predominant frequency. In a later development the work of Kolmogorov, Arnold and Moser (KAM) made it possible to construct quasi-periodic motions for Hamiltonian systems which again play a role in celestial mechanics. This construction can also be viewed as intimately related to the averaging method, although the corresponding existence proofs are much more complicated and subtle than in the case of periodic motions. It is natural to ask whether these results can be extended to general systems, just as Krylov-Bogoljubov's theory removed the Hamiltonian formalism from the averaging method. To do just this is, in principle, the motivation for the present book.

However, it has to be said at the outset that for general systems the quasi-periodic motions play a less significant role than periodic motions. There is a basic difference between Hamiltonian and dissipative systems. While there are open sets of Hamiltonian systems possessing quasi-periodic solutions such solutions usually disintegrate under general perturbations. In other words, in contrast to periodic solutions, the quasi-periodic solutions persist under perturbation only if the class of differential equation is sufficiently restricted. This shortcoming can be overcome by considering families of systems, depending on a number of parameters which are so determined as to force the existence of quasi-periodic solutions.

To make this remark more precise we describe the type of result to which this first chapter is devoted: the authors consider a system

$$\begin{aligned} \frac{dh}{dt} &= Hh + F(h, \varphi, \Delta) \\ \frac{d\varphi}{dt} &= \omega + \Delta + f(h, \varphi, \Delta) \end{aligned} \tag{1}$$

where $h = (h_1, h_2, \ldots, h_m)$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$, $\omega = (\omega_1, \ldots, \omega_n)$ are vectors, and H an m by m matrix. The vector functions F, f are assumed to be small and to have period 2π in each $\varphi_1, \varphi_2, \ldots, \varphi_n$, i.e., the latter are angular variables. Thus in the unperturbed situation, for $F = 0$, $f = 0$, the systems possess $h = 0$ as an invariant torus T_0^n, on which the solutions are given by $\varphi_v = \omega_v t + \varphi_v(0)$, if also $\Delta = 0$. Thus, if the ω_v are rationally independent numbers, these solutions are dense on the torus; they are called quasi-periodic solutions. Moreover, if the eigenvalues of the matrix H have negative real parts then all solutions near the torus T_0^n approach it exponentially, i.e., T_0^n is asymptotically stable.

The problem is to study this situation under small perturbation in analogy to the perturbation theory of periodic orbits. It is fairly clear that even after a small perturbation there exists an asymptotically stable torus T^n near T_0^n. This is in fact true, but the induced flow on T^n will in general not be quasi-periodic; for example, for $n = 2$, the flow on T^2 will in general possess finitely many periodic orbits which are approached by the other solutions on T^2 for $t \to +\infty$ or $t \to -\infty$. This phenomenon is called "entrainment of frequencies", or the "lock in" phenomenon.

How can we then have a perturbation theory for the quasi-periodic solutions of (1)? The answer is that we use the auxiliary parameters $\Delta = (\Delta_1, \ldots, \Delta_n)$ to control the frequencies of the perturbed system to have the frequencies ω independently of the perturbation. Thus, the result takes the form: under appropriate conditions on ω, H and for sufficiently small F, f one can determine the parameters Δ such that for this choice of Δ the system (1) has an asymptotically stable torus, on which all solutions are quasi-periodic with the same frequencies ω as the unperturbed system.

This result may sound somewhat artificial but, if one asks for a perturbation theory of quasi-periodic solutions, this lies in the nature of things. For Hamiltonian systems which are nearly integrable the additional parameters are automatically present; essentially they are the values of the integrals of the unperturbed system. This explains the remark made above that the quasi-periodic motions are more significant for Hamiltonian or reversible systems than for general systems where they occur rarely for an individual system and therefore one has to consider systems depending on several parameters.

The theory necessary to establish such quasi-periodic solution is formidable. Basically it involves the same apparatus as the previous KAM-theory, such as rapidly convergent iteration methods for real analytic systems and smoothing techniques for C^r-systems. The study of the linearized equations leads to an analogue of Floquet theory, i.e., linear systems of differential equations with quasi-periodic coefficients. The problem is to find coordinate transformations with quasi-periodic coefficients which brings the system into one with constant coefficients. Unfortunately, no general theory of this nature is available and therefore the book, Chapter 5, is concerned with perturbation results, and considers systems which are close to those with constant coefficients and eigenvalues not on the imaginary axis. The other chapters deal with many related questions about the perturbation theory of invariant tori, the linearized flow, the differential equations near such invariant tori. The technique is based on repeated coordinate transformations and rapid iteration methods. This approach is carried out repeatedly, in the frame work of real analytic functions and of C^r-functions.

(II)This may suffice for describing the content and motivation of this book which could be summarized by this phrase: small divisor problems for non-Hamiltonian systems of differential equations–and we turn to the way in which this project is carried out. It is a difficult task to present such complicated and technical material and to the reviewer the exposition is not satisfactory, but appears tiresome, repetitive and monotonous. This book is a translation of a Russian book which appeared in 1969. Actually Chapter 1 is a slightly edited version of 8 lectures given by Bogoljubov in June/July 1963 at a summer school in Kanev. These lectures were published Kiev 1964 but are hardly accessible in the West. The bibliography is, aside from the alphabetical ordering, the same as in the Russian original, making the references obsolete. The appendix, 18 pages, added to the translation contains some references to the recent literature, but this does not suffice to remedy this defect. On the other hand, another reference is certainly misleading: in paragraph 16 Denjoy's theory concerning diffeomorphisms of the circle is discussed. As is well known, this theory establishes that any sufficiently smooth circle mapping without periodic points can be mapped by a homeomorphism, say h, into a rotation. It has been an open question for a long time to find conditions on the rotation number which lead to smooth homeomorphisms h. This question was solved only in 1976 in the remarkable work of R. Herman (Thése, Orsay 1976; for an excellent exposition, see P. Deligne, Seminaire Bourbaki 477, 1976) which appeared too late to become known to the authors. However, the reference to the work of Finzi **[20]** on p. 101 is misleading, since his proof is simply wrong. A footnote on the same page stating "the validity of this statement remains an open question because the derivations in [20] are yet to be proved conclusively" does not really clarify the situation.

This brings us to another shortcoming, the translation from the Russian to the English. We illustrate the low quality with few examples (no completeness intended). The name of the second author is translated as Mitropoliskii, although his name is commonly known in the Western literature as Mitropolski, the soft sign being mute. On p. 273 one finds "column" translated as "colon"! On p. 123 the paragraph after the statement of Theorem 14 begins with the word "Proof", although what follows is merely a remark. The actual proof is given only in the following section (21). Incidentally, this error is *not* present in the Russian original. Also on p. 96 the second formula is garbled, although the Russian book has it correctly. "Bendixson" appears as "Bendizon," "J. Nash" as "J. Nach", etc. One wonders whether the authors had the opportunity to proofread the English version of their book.

The names of the authors of this monograph bring to mind the well-known book *Asymptotic methods in the theory of nonlinear oscillations* by Bogoljubov and Mitropolski, but one has to be aware that these two works are of different character. While the old book discusses various asymptotic methods, in particular the averaging methods, with emphasis on applications to various interesting real problems the present book is of a technical nature. Maybe the only common feature is the translation by the Hindustan Publishing Company (India) which is equally poor. If one wants to learn about the delicate existence proofs for small divisor problems one still does well to read the excellent papers in the Uspehi Mat. Nauk by V. I. Arnold.

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