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REAL ALGEBRAIC VARIETY STRUCTURES ON P. L. MANIFOLDS

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A closed smooth manifold M^m is said to bound a smooth *spine-manifold* if M bounds a compact smooth manifold W^{m+1} and if there are a finite number of transversally intersecting closed submanifolds $\{M_i\}$ of W such that $W/\bigcup M_i \approx \text{cone}(M)$, where $\approx$ is piecewise differentiable homeomorphism.

DEFINITION. An A_1-structure on a P. L. manifold M^m is: $M = M_0 \cup \bigcup_i \text{cone}(\Sigma_i) \times N_i$ where M_0 is a codimension zero smooth submanifold of M, $\partial M_0 = \amalg_i \Sigma_i \times N_i$, N_i's are smooth manifolds and Σ_i's are exotic spheres bounding smooth spine-manifolds.

A_1-structures satisfy regular neighborhood and product structure properties, and there is a classifying space B_{A_1} with inclusions $B_0 \to B_{A_1} \to B_{PL}$ (see [3]). This reduces the existence of A_1-structure on a P. L. manifold to a bundle lifting problem.

THEOREM 1. *Any closed A_1-manifold is P. L. homeomorphic to a real algebraic variety.*

COROLLARY 1. *All P. L. manifolds of dimension less than* 10 *are P. L. homeomorphic to real algebraic varieties* (*also see* [1]).

THEOREM 2. *If a closed smooth manifold bounds a smooth spine-manifold, then it can be represented as a link of an isolated real algebraic singularity.* (*Converse of this is the Hironaka's resolution theorem.*)

COROLLARY 2. *Elements of* Γ_8, $2\Gamma_{10}$, *and all exotic spheres which admit fixed point free smooth involutions are links of real algebraic singularities* (*also see* [2]).

A BRIEF SKETCH OF THE PROOFS. Let M be a closed A_1-manifold. For simplicity assume $M^m = M_0^m \cup \text{cone}(\Sigma^{m-1})$; then there is W^m with closed submanifolds $\{M_i\}$ such that $W/\bigcup M_i \approx \text{cone}(M)$, and $\partial W = \Sigma$. Let $\widetilde{M} = M_0 \cup W$.

By proving a relative version of the Nash-Tognoli approximation theorem we can make the smooth manifold $\widetilde{M}$ a real algebraic variety V, so that the smooth submanifolds $\{M_i\}$ of $\widetilde{M}$ correspond to the subvarieties $\{V_i\}$ of V.

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Let $V = f^{-1}(0)$ and $\bigcup V_i = g^{-1}(0)$, where $f(x)$ and $g(x)$ are polynomials. Let $F(x, t) = f(x)^2 + (tg(x) - 1)^2$, then $\hat{F}(y) = |y|^{2d}F(y/|y|^2) = 0$, $y = (x, t)$, $d =$ degree F, gives the equations of

$$V/\bigcup V_i \approx \widetilde{M}/\bigcup M_i = M_0 \cup (W/\bigcup M_i) \approx M_0 \cup \operatorname{cone}(\Sigma) = M.$$

This sketches the idea of the proofs of Theorem 1 and 2. Corollary 1 and 2 are true because elements of Γ_8, $2\Gamma_{10}$ bound spine manifolds (see [4]); and any exotic sphere Σ with fixed point free smooth involution τ bounds the obvious spine manifold $\Sigma \times I/(x, 0) \sim (\tau(x), 0)$.

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