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AUTOMORPHISMS OF SEMIGROUPS OF COMPLEXES OF ABELIAN GROUPS

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In the study of any algebraic system, one of the first objects investigated is its automorphism group [1]. The automorphisms of groups have been investigated so extensively that we make no attempt at a list of references. In [4], A. R. Richardson studied the automorphisms of groupoids. Semigroups of finite complexes in groups are central to the study of retractable groups [2], and here we announce some properties of the automorphism group of such a semigroup in the case in which the underlying group is abelian. In particular, when the underlying group is cyclic we classify these automorphism groups.

If G is a group, then the collection $F(G)$ of all finite nonempty subsets of G is a semigroup, where $AB = \{ab \mid a \in A \text{ and } b \in B\}$. Each automorphism α of G induces an automorphism α^* of $F(G)$ where $A\alpha^* = \{a\alpha \mid a \in A\}$. An automorphism of $F(G)$ of this type will be called a *standard automorphism* of $F(G)$.

THEOREM 1. *If φ is an automorphism of $F(G)$, then φ is a standard automorphism if and only if φ is inclusion preserving.*

A homomorphism σ of $F(G)$ into G such that $\{g\}\sigma = g$ for every g in G is called a *retraction* of G. A group G is called *retractable* if it admits a retraction. The concept of a retractable group was introduced in [2] where it was shown that the class of retractable groups is a proper subclass of the class of torsion free groups and the class of lattice-ordered groups is a proper subclass of the class of retractable groups. Hence, the class of torsion free abelian groups is a proper subclass of the class of retractable groups. "It seems to be a rather difficult problem to determine all abelian groups with commutative endomorphism ring" [3, p. 205]. If G is a torsion free abelian group then it is easy to show that the automorphism group of $F(G)$ is nonabelian.

THEOREM 2. *If G is an abelian group, σ is a retraction of G, and φ_σ is given by*

$$A\varphi_\sigma = (A\sigma)A(A^{-1}\sigma)$$

for every $A \in F(G)$, then φ_σ is an automorphism of $F(G)$. Moreover,

(i) if *φ_σ is not the identity automorphism, then φ_σ is a nonstandard automorphism of $F(G)$ of infinite order*;

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(ii) *the automorphism group of $F(G)$ is an infinite nonabelian group.*

Let Z denote the additive group of integers. In [2] the collection of retractions of Z was completely determined. If $k \in Z$ and if σ_k is defined by $A\sigma_k = (k + 1)\max A - k \min A$ for all $A \in F(Z)$, then $\{\sigma_k | k \in Z\}$ is the collection of retractions of Z. We have shown that the automorphisms of $F(Z)$ that are induced by retractions of Z are contained in the cyclic subgroup generated by $\varphi_{\sigma_{-1}}$. With this information we are able to show

THEOREM 3. *The automorphism group of $F(Z)$ is isomorphic to a nonabelian splitting extension of the integers by the Klein four-group.*

If n is a natural number, let Z_n denote the group of integers modulo n. Clearly, the automorphism groups of $F(Z_1)$ and $F(Z_2)$ are trivial. It can be shown that $F(Z_3)$, $F(Z_4)$, and $F(Z_5)$ admit nonstandard automorphisms.

THEOREM 4. *If n is a natural number, $n \geqslant 6$, then $F(Z_n)$ admits only standard automorphisms and hence the automorphism group of $F(Z_n)$ is isomorphic to the group of automorphisms of Z_n.*

The proofs of the above results, as well as others, are computational and will appear elsewhere.

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