BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

FAILURE OF A QUADRATIC ANALOGUE OF SERRE'S CONJECTURE

BY S. PARIMALA

Communicated by Olga Taussky Todd, July 30, 1976

Let A be a commutative ring with identity. By an *inner product A-space* we shall understand, as in [6], a pair (P, q), where P is a finitely generated projective A-module and q is a symmetric bilinear form $P \times P \longrightarrow A$ which is nonsingular (i.e. induces an isomorphism $P \xrightarrow{\sim} P^*$). If B is a commutative A-algebra we obtain an inner product B-space $(B \otimes_A P, B \otimes_A q)$. Inner product B-spaces isomorphic to one of these will be said to be *extended* from A.

The quadratic analogue of Serre's conjecture is the affirmation of:

Suppose A is a polynomial algebra $K[X_1, \ldots, X_n]$ over a field K. Is every inner product A-space extended from K?

This question is motivated by the following evidence.

(1) Serre's conjecture that projective A-modules are free, hence extended from K, has recently been proved by Quillen and Suslin (cf. [4]). Moreover this immediately implies that "symplectic A-spaces" are extended from K (see e.g. [1, Chapter IV, (4.11.2)]).

(2) If $\mathrm{Char}(K) \neq 2$ then a theorem of Karoubi [7, Theorem 1.1] implies that every inner product A-space is *stably isomorphic* to one extended from K.

(3) A theorem of Harder (see [8, Theorem 13.4.3]) gives an affirmative response to (QS) for $n = 1$.

A major tool in Quillen's proof of Serre's conjecture is:

QUILLEN'S LOCALIZATION THEOREM [11]. *Let A be a commutative ring, let T be an indeterminate, and let M be a finitely presented $A[T]$-module. If, for all maximal ideals $\mathfrak{m}$ of A, $M_\mathfrak{m}$ is extended from $A_\mathfrak{m}$, then M is extended from A.*

(4) The analogue of Quillen's localization theorem for inner product spaces has been proved in [3].

The other main tool Quillen uses is:

HORROCK'S THEOREM [5]. *Let A be a local ring and let P be a finitely generated projective $A[T]$-module. If P extends to a locally free sheaf on $\mathbf{P}^1_A$, then P is extended from A (hence free).*

AMS (MOS) subject classifications (1970). Primary 15A63; Secondary 18F25.

It is natural then to ask:

(QH) *Is the analogue of Horrock's theorem for inner product spaces valid*?

We here answer (QS) *and* (QH) *negatively, with the following example.* Let $A = \mathbf{R}[X, Y]$. Consider the symmetric 4×4 matrix over

$$A, S = \begin{pmatrix} \alpha & \beta \\ {}^t\beta & \alpha \end{pmatrix}$$

where

$$\alpha = \begin{pmatrix} 4 + Y^2(1 + X^2) & XY(1 + Y^2) \\ XY(1 + Y^2) & 1 + X^2Y^4 \end{pmatrix} = {}^t\alpha,$$

(t denotes transpose), and

$$\beta = \begin{pmatrix} 0 & Y(1 + X^2Y^2) \\ -Y(1 + X^2Y^2) & 0 \end{pmatrix} = -{}^t\beta.$$

Let q be the bilinear form on $P = A^4$ with matrix S relative to the natural basis of A^4.

THEOREM. (1) *(P, q) is an inner product space over $A = \mathbf{R}[X, Y]$ which is not extended from* $\mathbf{R}$.

(2) *For each prime ideal $\mathfrak{p}$ of A, the inner product $A_{\mathfrak{p}}$-space $(P_{\mathfrak{p}}, q_{\mathfrak{p}})$ is extended from* $\mathbf{R}$.

(3) *(P, q) extends to a sheaf of inner product spaces over $\mathbf{P}^1_{\mathbf{R}[X]}$, yet for some prime ideal $\mathfrak{p}$ of $\mathbf{R}[X]$, the inner product $\mathbf{R}[X]_{\mathfrak{p}}[Y]$-space $(P_{\mathfrak{p}}, q_{\mathfrak{p}})$ is not extended from $\mathbf{R}[X]_{\mathfrak{p}}$.*

REMARKS. (a). The matrix S is derived from the hermitian matrix $H = \alpha + i\beta$ over $\mathbf{C}[X, Y]$, which was discovered from an investigation of the classification of the rank 1 projective $\mathbf{H}[X, Y]$-modules ($\mathbf{H}$ = quaternions) in terms of hermitian matrices, established in [9], [10]. The analogue of the above theorem for the hermitian $\mathbf{C}[X, Y]$-space defined by H is also valid.

(b) The matrix S has entries in $\mathbf{Z}[X, Y]$, and $\det(S) = 16$. Thus (P, q) is extended from an inner product space over $\mathbf{Z}[1/2]\ [X, Y]$.

(c) If one considers quadratic forms rather than symmetric bilinear forms the analogue of (QS) has a negative response (in characteristic 2 of course) al-already for $n = 1$ (see [7, p. 318]).

(d) Bass [2] has investigated (QS) when K is algebraically closed.

(e) It follows from Harder's theorem (3) that the answer to (QH) is trivially in the affirmative if A is a field. Our theorem shows that the answer is negative already for the discrete valuation ring $\mathbf{R}[X]_{\mathfrak{p}}$.

(f) The proof of the above theorem will appear elsewhere.

I am thankful to Professor H. Bass for carefully going through my calculations and his great help in preparing this manuscript.

I am more than grateful to Professor R. Sridharan to whom this work is respectfully dedicated.

REFERENCES

1. H. Bass, *Unitary algebraic K-theory*, Algebraic K-theory. III (Proc. Conf., Seattle Res. Center, Battelle Memorial Inst., 1972), Lecture Notes in Math., vol. 343, Springer-Verlag, Berlin and New York, 1973, pp. 57–265. MR 51 #8211.

2. ———, *Quadratic modules over polynomial rings* (to appear).

3. H. Bass, E. H. Connell, and D. L. Wright, *Locally polynomial algebras are symmetric algebras* (to appear).

4. D. Ferrand, *Les modules projectifs de type fini sur un anneau de polynômes sur un corps sont libres*, Séminaire Bourbaki, Exposé 484, Juin 1976.

5. G. Horrocks, *Projective modules over an extension of a local ring*, Proc. London Math. Soc. (3) 14 (1964), 714–718. MR 30 #121.

6. D. Husemoller and J. Milnor, *Symmetric bilinear forms*, Ergebnisse der Mathematik, Band 73, Springer-Verlag, Berlin and New York, 1973.

7. M. Karoubi, *Périodicité de la K-théorie hermitienne*, Algebraic K-theory. III (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972), Lecture Notes in Math. vol. 343, Springer-Verlag, Berlin and New York, 1973, pp. 301–411. MR 48 #3656c.

8. M. Knebusch, *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. 1969/70, 93–157. MR 42 #6001.

9. S. Parimala, *Projective modules and hermitian matrices*, J. Pure Applied Algebra 7 (1976), 5–14.

10. S. Parimala and R. Sridharan, *Projective modules over polynomial rings over division rings*, J. Math. Kyoto Univ. 15 (1975), 129–148. MR 51 12935.

11. D. Quillen, *Projective modules over polynomial rings*, Invent. Math. (to appear).

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY–5, INDIA